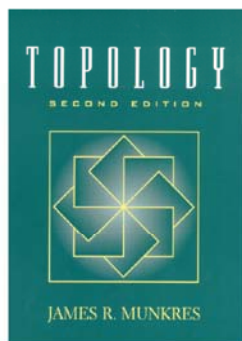


Introduction to Topology

Chapter 9. The Fundamental Group

Section 54 The Fundamental Group of the Circle—Proofs of Theorems



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Lemma 54.1

Lemma 54.1 Continued

First, define $\tilde{f}(0) = e_0$. We now inductively define the lifting \tilde{f} . The set $f([s_i, s_{i+1}])$ lies in some open U that is evenly covered by p .

Let $\{V_2\}$ be a partition of $p^{-1}(U)$ into slices. Then (by definition) each V_2 is mapped homeomorphically onto U by p .

Suppose \tilde{f} has been defined on $[s_0, s_i]$. Then $f(s_i)$ lies in some open V_2 , say in V_i .

Define $\tilde{f}(s)$ for $s \in [s_i, s_{i+1}]$ by the equation

$$\tilde{f}(s) = (p|_{V_i})^{-1}(f(s)) \quad (1)$$

(since p is a homeomorphism on V_i , then it is one to one and so $p|_{V_i}$ is [illegible]). Since $p|_{V_i}$ is a homeomorphism, then its inverse is continuous and (since f is continuous — it is a path) then \tilde{f} is continuous on $[s_i, s_{i+1}]$.

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Lemma 54.1

Lemma 54.1

Lemma 54.1. Let $p : E \rightarrow B$ be a covering map, and let $p(e_0) = b_0$. Any path $f : [0, 1] \rightarrow B$ beginning at b_0 has a unique lifting to a path \tilde{f} in E beginning at e_0 .

Proof. Cover B by open sets U each of which is evenly covered by p (which can be done by the definition of a covering map). Partition $[0, 1]$, say as $0 < s_0 < s_1 < \dots < s_n = 1$ such that for each i the set $f([s_i, s_{i+1}])$ lies entirely inside one of the open sets U . (Since f is continuous and $[0, 1]$ is compact, then $f([0, 1])$ is compact and so has a finite subcover of the open sets U .)

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Lemma 54.1

Lemma 54.1 Continued

Continue in this way to define \tilde{f} on all of $[0, 1]$. Then by construction, \tilde{f} is continuous on $[0, 1]$. Also,

$$\begin{aligned} p \circ \tilde{f}(s) &= p \circ ((p|_{V_i})^{-1}(f(s))) \text{ for } s \in [s_i, s_{i+1}] \\ &= p(p|_{V_i})^{-1}(f(s)) = f(s) \end{aligned} \quad (2)$$

and so $p \circ \tilde{f} = f$ on $[0, 1]$.

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Lemma 54.1 Continued

Now we show that \tilde{f} as defined above is unique. Suppose that \tilde{f}' is another lifting of f beginning at e_0 . Then $\tilde{f}'(0) = e_0 = \tilde{f}(0)$.

Suppose that $\tilde{f}'(s) = \tilde{f}(s)$ for all s such that $0 \leq s \leq s_i$. Let V_i be the V_2 containing $f(s_i)$ as above. For $s \in [s_i, s_{i+1}]$, $\tilde{f}'(s)$ is defined as $(p|_{V_i})^{-1}(f(s))$.

Since \tilde{f}' is a lifting of f then by definition $p \circ \tilde{f}' = f$ and so \tilde{f}' must carry the interval $[s_i, s_{i+1}]$ into the set $p^{-1}(U)$ where U is some open set containing $f([s_i, s_{i+1}])$, as above.

Since U is evenly covered by p (as above) then $p^{-1}(U) = UV_2$ where V_2 are the slices. Now the V_2 are open and disjoint and $\tilde{f}'([s_i, s_{i+1}])$ is connected (continuous function applied to connected set) then it must lie entirely in one of the V_2 (otherwise the multiple V_2 containing it form a separation of it, a contradiction).

Theorem 54.3

Theorem 54.3. Let $p : E \rightarrow B$ be a covering map, and let $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1 . Let \tilde{f}, \tilde{g} be their respective liftings to paths in E beginning at e_0 . If f and g are path homotopic, then \tilde{f} and \tilde{g} end at the same point of E and are path homotopic.

Proof. Let $F : I \times I \rightarrow B$ be a hypothesized homotopy between f and g . Then $F(0, 0) = b_0$. Let $\tilde{F} : I \times I \rightarrow E$ be the lifting of F to E such that $\tilde{F}(0, 0) = e_0$ (existence and uniqueness of \tilde{F} is given by Lemma 54.2).

By Lemma 54.2 \tilde{F} is a path homotopy since F is, and so $\tilde{F}(\{0\} \times I) = \{e_0\}$ and $\tilde{F}(\{1\} \times I)$ is a singleton, say $\{e_1\}$.

The restriction $\tilde{F}|_{I \times \{0\}}$ of \tilde{F} to the bottom edge of $I \times I$ is a path in E beginning at e_0 (since \tilde{F} is continuous). that is a lifting of $\tilde{F}|_{I \times \{0\}}$.

Lemma 54.1 Continued

Because $\tilde{f}'(s) = \tilde{f}(s_i) \in V_i$, then all of $\tilde{f}'([s_i, s_{i+1}])$ must lie in V_i must lie in V_i . Thus for each $s^* \in [s_i, s_{i+1}]$, $\tilde{f}'(s^*)$ must equal some point $y \in V_i$.

But p is a homeomorphism of V_i with U and there is only one point of V_i which is mapped to $f(s^*)$ by p . Since $(p|_{V_i})^{-1}f(s^*)$ is one such point, then y must equal this point:

$$y = \tilde{f}'(s^*) = (p|_{V_i})^{-1}(f(s^*)) \quad (3)$$

and

$$p(y) = p \circ \tilde{f}'(s^*) = (p \circ (p|_{V_i})^{-1})f(s^*) \quad (4)$$

Or $(p \circ \tilde{f}'(s^*)) = f(s^*)$ for all $s^* \in [s_i, s_{i+1}]$. So $(p \circ \tilde{f}')(s) = (p \circ \tilde{f})(s)$ for all $s \in [0, 1]$. Therefore, since p is one to one, $\tilde{f}'(s) = \tilde{f}(s)$ for all $s \in [0, 1]$. That is, $\tilde{f}' = \tilde{f}$ and \tilde{f} is unique. \square

Theorem 54.3

By the uniqueness of path liftings in Lemma 54.1, we must have $\tilde{F}(s, 0) = \tilde{f}(s)$ (since \tilde{f} is a lifting of f). Similarly, $\tilde{F}|_{I \times \{1\}}$ is a path in E that is a lifting of $F|_{I \times \{1\}}$ and it begins at e_0 because $\tilde{F}(\{0\} \times I) = \{e_0\}$; and uniqueness of path liftings in Lemma 54.1 gives that $\tilde{F}(s, 1) = \tilde{g}(s)$.

So both \tilde{f} and \tilde{g} end at $\tilde{F}(\{1\} \times I) = \{e_1\}$, therefore \tilde{F} is a path homotopy between \tilde{f} and \tilde{g} . \square

Theorem 54.4

Theorem 54.4. Let $p : E \rightarrow B$ be a covering map, and let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence

$$\phi : \pi(B, b_0) \rightarrow p^{-1}(b_0) \quad (5)$$

is surjective (onto). If E is simply connected, it is bijective.

Proof. If E is path connected, then, given any $e_1 \in p^{-1}(b_0)$, there is a path \tilde{f} in E from e_0 to e_1 . Define $f = p \circ \tilde{f}$ and then $f(0) = p \circ \tilde{f}(0) = p(e_0) = b_0$ and $f(1) = p \circ \tilde{f}(1) = p(e_1) = b_0$ since $e_1 \in p^{-1}(b_0)$.

So f is a loop in B at b_0 and so

$$\phi([f]) = \tilde{f}(1) = e_1, \text{ so } \phi \text{ is onto } p^{-1}(b_0). \quad (6)$$

Theorem 54.4 Continued

Since p is continuous then $p \circ \tilde{F}$ is continuous and hence $p \circ \tilde{F}$ is a homotopy of f with g . Therefore, $[f] = [g]$ and ϕ is one to one.

Since simply connected spaces are path connected [REF], then the first paragraph gives that ϕ is onto, so ϕ is bijective on a simply connected space. \square

Theorem 54.4 Continued

Suppose E is simply connected. Let $[f]$ and $[g]$ be two elements of $\pi_1(B, b_0)$ such that $\phi([f]) = \phi([g])$. Let \tilde{f} and \tilde{g} be the liftings of f and g respectively to paths in E that begin at e_0 . Since $\phi([f]) = \phi([g])$, the endpoints of f and g are the same, so $f(1) = g(1)$ and hence $\tilde{f}(1) = \tilde{g}(1)$. So \tilde{f} and \tilde{g} have the same initial and final points.

Since E is simply connected, there is a path homotopy \tilde{F} in E between \tilde{f} and \tilde{g} . Since $\tilde{F}(I \times \{0\}) = \tilde{f}$ and $\tilde{F}(I \times \{1\}) = \tilde{g}$ then

$$\begin{aligned} p \circ \tilde{F}(I \times \{0\}) &= p \circ \tilde{f} = f \text{ and} \\ p \circ \tilde{F}(I \times \{1\}) &= p \circ \tilde{g} = g \end{aligned} \quad (7)$$

since \tilde{f} and \tilde{g} are liftings of f and g , respectively.

Theorem 54.5

Theorem 54.5. The fundamental group of S^1 is isomorphic to the additive group of integers, \mathbb{Z} .

Proof. Let $p : \mathbb{R} \rightarrow S^1$ be the covering map of Theorem 53.1:

$$p(s) = (\cos(2\pi s), \sin(2\pi s)) \quad (8)$$

Let $e_0 = 0$ and $b_0 = p(e_0) = (1, 0)$. Then $p^{-1}(b_0) = \mathbb{Z}$. Since \mathbb{R} is simply connected, the lifting correspondence $\phi : \pi_1(S^1, b_0) \rightarrow \mathbb{Z}$ is bijective by Theorem 54.4. We now show that ϕ is a group homomorphism.

Given $[f]$ and $[g]$ in $\pi_1(B, b_0)$, let \tilde{f} and \tilde{g} be liftings of f and g , respectively, to paths in \mathbb{R} beginning at 0. We know by the definition of ϕ that $\phi([f]), \phi([g]) \in \mathbb{Z}$ and are the endpoints in \mathbb{R} of $\tilde{f}(1)$ and $\tilde{g}(1)$, respectively. Say $n = \phi([f])$ and $m = \phi([g])$. Let \tilde{g}' be the path $\tilde{g}'(s) = n + \tilde{g}(s)$ in \mathbb{R} (so \tilde{g}' begins at n and ends at $n + \tilde{g}(1) = n + m$).

Theorem 54.5 Continued

Because $p(n+x) = p(x)$ (p is a periodic function with period 1) for all $x \in \mathbb{R}$, the path \tilde{g}' is a lifting of g which begins at n , since $p \circ \tilde{g}' = g$ on I . Since \tilde{f} ends at n , then $\tilde{f} * \tilde{g}'$ is defined and it is a lifting of $f * g$ beginning at 0, since

$$p(\tilde{f} * \tilde{g}') = p(\tilde{f}) * p(\tilde{g}') = f * g \quad (9)$$

The end point of $f * g$ is $n + m$ and the end point of \tilde{g}' is $n + m$ (we need to introduce \tilde{g}' because $\tilde{f} * \tilde{g}$ is not defined — \tilde{f} ends at n and \tilde{g} begins at 0). So

$$\phi([f * g]) = \phi([\tilde{f}] * [\tilde{g}]) = m + n \quad (10)$$

since $\tilde{f} * \tilde{g}'$ ends at $m + n$ and

$$\phi([\tilde{f}]) + \phi([\tilde{g}]) = n + m \quad (11)$$

Since \tilde{f} ends at n and \tilde{g} ends at m .

Theorem 54.5 Continued

Therefore,

$$\phi([f] * [g]) = \phi([f]) + \phi([g]) \quad (12)$$

and ϕ is a group homomorphism.

Now if $\phi : G \rightarrow G'$ is a group homomorphism, then the set $\phi(G)$ is a subgroup of G' (see Fraleigh, Theorem 13.12(3)). Every subgroup of \mathbb{Z} is isomorphic to \mathbb{Z} (see Fraleigh's Corollary 6.7 and Theorem 11.12) so ϕ is an group isomorphism between $\pi_1(B, b_0)$ and \mathbb{Z} . \square