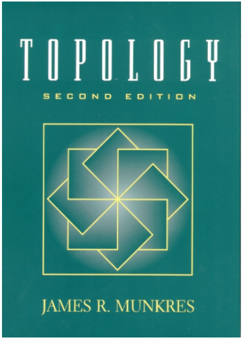
# Introduction to Topology

**Chapter 9. The Fundamental Group**

Section 54 The Fundamental Group of the Circle—Proofs of Theorems



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# Lemma 54.1

**Lemma 54.1.** Let $p : E \to B$ be a covering map, and let $p(e_0) = b_0$. Any path $f : [0,1] \to B$ beginning at $b_0$ has a unique lifting to a path $f$ in $E$ beginning at $e_0$.

**Proof.** Cover $B$ by open sets $U$ each of which is evenly covered by $p$ ( which can be done by the definition of a covering map). Partition $[0,1]$, say as $0 < s_0 < s_1 < \ldots < s_n = 1$ such that for each $i$ the set $f([s_i, s_{i+1}])$ lies entirely inside one of the open sets $U$. (Since $f$ is continuous and $[0,1]$ is compact, then $f([0,1])$ is compact and so has a finite subcover of the open sets $U$.)

# Lemma 54.1 Continued

First, define $\tilde{f}(0) = e_0$. We now inductively define the lifting $\tilde{f}$. The set $f([s_i, s_{i+1}])$ lies in some open $U$ that is evenly covered by $p$.

Let $\{V_2\}$ be a partition of $p^{-1}(U)$ into slices. Then (by definition) each $V_2$ is mapped homeomorphically onto $U$ by $p$.

# Lemma 54.1 Continued

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Suppose $\tilde{f}$ has been defined on $[s_o, s_i]$. Then $f(s_i)$ lies in some open $V_2$, say in $V_i$.

# Lemma 54.1 Continued

First, define $\tilde{f}(0) = e_0$. We now inductively define the lifting $\tilde{f}$. The set $f([s_i, s_{i+1}])$ lies in some open $U$ that is evenly covered by $p$.

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Suppose $\tilde{f}$ has been defined on $[s_o, s_i]$. Then $f(s_i)$ lies in some open $V_2$, say in $V_i$.

Define $\tilde{f}(s)$ for $s \in [s_i, s_{i+1}]$ by the equation

$$\tilde{f}(s) = (p|_{V_i})^{-1}(f(s)) \quad (1)$$

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First, define $\tilde{f}(0) = e_0$. We now inductively define the lifting $\tilde{f}$. The set $f([s_i, s_{i+1}])$ lies in some open $U$ that is evenly covered by $p$.

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Define $\tilde{f}(s)$ for $s \in [s_i, s_{i+1}]$ by the equation

$$\tilde{f}(s) = (p|_{V_i})^{-1}(f(s)) \quad (1)$$

(since $p$ is a homeomorphism on $V_i$, then it is one to one and so $p|_{V_i}$ is [illegible]). Since $p|_{V_i}$ is a homeomorphism, then its inverse is continuous and (since $f$ is continuous — it is a path) then $\tilde{f}$ is continuous on $[s_i, s_{i+1}]$.

# Lemma 54.1 Continued

First, define $\tilde{f}(0) = e_0$. We now inductively define the lifting $\tilde{f}$. The set $f([s_i, s_{i+1}])$ lies in some open $U$ that is evenly covered by $p$.

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# Lemma 54.1 Continued

Continue in this way to define $\tilde{f}$ on all of $[0,1]$. Then by construction, $\tilde{f}$ is continuous on $[0,1]$. Also,

$$\begin{aligned} p \circ \tilde{f}(s) &= p \circ ((p|_{V_i})^{-1}(f(s)) \textit{ for } s \in [s_i, s_{i+1}] \\ &= p(p|_{V_i})^{-1}(f(s)) = f(s) \end{aligned} \tag{2}$$

and so $p \circ \tilde{f} = f$ on $[0,1]$.

# Lemma 54.1 Continued

Now we show that $\tilde{f}$ as defined above is unique. Suppose that $\tilde{f}'$ is another lifting of [illegible] beginning at $e_0$. Then $\tilde{f}'(0) = e_0 = \tilde{f}(0)$.

Suppose that $\tilde{f}'(s) = \tilde{f}(s)$ for all $s$ such that $0 \leq s \leq s_i$. Let $V_i$ be the $V_2$ containing $f(s_i)$ as above. For $s \in [s_i, s_{i+1}]$, $\tilde{f}'(s)$ is defined as $(p|_{V_i})^{-1}(f(s))$.

# Lemma 54.1 Continued

Now we show that $\tilde{f}$ as defined above is unique. Suppose that $\tilde{f}'$ is another lifting of [illegible] beginning at $e_0$. Then $\tilde{f}'(0) = e_0 = \tilde{f}(0)$.

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Since $\tilde{f}'$ is a lifting of $f$ then by definition $p \circ \tilde{f}' = f$ and so $\tilde{f}'$ must carry the interval $[s_i, s_{i+1}]$ into the set $p^{-1}(U)$ where $U$ is some open set containing $f([s_i, s_{i+1}])$, as above.

# Lemma 54.1 Continued

Now we show that $\tilde{f}$ as defined above is unique. Suppose that $\tilde{f}'$ is another lifting of [illegible] beginning at $e_0$. Then $\tilde{f}'(0) = e_0 = \tilde{f}(0)$.

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Since $U$ is evenly covered by $p$ (as above) then $p^{-1}(U) = UV_2$ where $V_2$ are the slices. Now the $V_2$ are open and disjoint and $\tilde{f}'([s_i, s_{i+1}])$ is connected (continuous function applied to connected set) then it must lie entirely in one of the $V_2$ (otherwise the multiple $V_2$ containing it form a separation of it, a contradiction).

# Lemma 54.1 Continued

Now we show that $\tilde{f}$ as defined above is unique. Suppose that $\tilde{f}'$ is another lifting of [illegible] beginning at $e_0$. Then $\tilde{f}'(0) = e_0 = \tilde{f}(0)$.

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# Lemma 54.1 Continued

Because $\tilde{f}'(s) = \tilde{f}(s_i) \in V_i$, then all of $\tilde{f}'([s_i, s_{i+1}])$ must lie in $V_i$ must lie in $V_i$. Thus for each $s^* \in [s_i, s_{i+1}]$, $\tilde{f}'(s^*)$ must equal some point $y \in V_i$.

But $p$ is a homeomorphism of $V_i$ with $U$ and there is only one point of $V_i$ which is mapped to $f(s^*)$ by $p$. Since $(p|_{V_i})^{-1}f(s^*)$ is one such point, then $y$ must equal this point:

$$y = \tilde{f}'(s^*) = (p|_{V_i})^{-1}(f(s^*)) \tag{3}$$

and

$$p(y) = p \circ \tilde{f}'(s^*) = (p \circ (p|_{V_i})^{-1})f(s^*) \tag{4}$$

# Lemma 54.1 Continued

Because $\tilde{f}'(s) = \tilde{f}(s_i) \in V_i$, then all of $\tilde{f}'([s_i, s_{i+1}])$ must lie in $V_i$ must lie in $V_i$. Thus for each $s^* \in [s_i, s_{i+1}]$, $\tilde{f}'(s^*)$ must equal some point $y \in V_i$.

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Or $(p \circ \tilde{f}'(s^*)) = f(s^*)$ for all $s^* \in [s_i, s_{i+1}]$. So $(p \circ \tilde{f}')(s) = (p \circ \tilde{f})(s)$ for all $s \in [0, 1]$. Therefore, since $p$ is one to one, $\tilde{f}'(s) = \tilde{f}(s)$ for all $s \in [0, 1]$. That is, $\tilde{f}' = \tilde{f}$ and $\tilde{f}$ is unique. □

# Lemma 54.1 Continued

Because $\tilde{f}'(s) = \tilde{f}(s_i) \in V_i$, then all of $\tilde{f}'([s_i, s_{i+1}])$ must lie in $V_i$ must lie in $V_i$. Thus for each $s^* \in [s_i, s_{i+1}]$, $\tilde{f}'(s^*)$ must equal some point $y \in V_i$.

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# Theorem 54.3

**Theorem 54.3.** Let $p : E \to B$ be a covering map, and let $p(e_0) = b_0$. Let $f$ and $g$ be two paths in $B$ from $b_0$ to $b_1$. Let $\overline{f}$ $\overline{g}$ be their respective liftings to paths in $E$ beginning at $e_0$. If $f$ and $g$ are path homotopic, then $\overline{f}$ and $\overline{g}$ end at the same point of $E$ and are path homotopic.

**Proof.** Let $F : I \times I \to B$ be a hypothesized homotopy between $f$ and $g$. Then $F(0,0) = b_0$. Let $\tilde{F} : I \times I \to E$ be the lifting of $F$ to $E$ such that $\tilde{F}(0,0) = e_0$ (existence and uniqueness of $\tilde{F}$ is given by Lemma 54.2).

By Lemma 54.2 $\tilde{F}$ is a path homotopy since $F$ is, and so $\tilde{F}(\{0\} \times I) = \{e_0\}$ and $\tilde{F}(\{1\} \times I)$ is a singleton, say $\{e_1\}$. The restriction $\tilde{F}|_{I\times\{0\}}$ of $\tilde{F}$ to the bottom edge of $I \times I$ is a path in $E$ beginning at $e_0$ (since $\tilde{F}$ is continuous). that is a lifting of $\tilde{F}|_{I\times\{0\}}$.

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By the uniqueness of path liftings in Lemma 54.1, we must have $\tilde{F}(s,0) = \tilde{f}(s)$ (since $\tilde{f}$ is a lifting of $f$). Similarly, $\tilde{F}|_{I\times\{1\}}$ is a path in $E$ that is a lifting of $F|_{I\times\{1\}}$ and it begins at $e_0$ because $\tilde{F}(\{0\}\times I) = \{e_0\}$; and uniqueness of path liftings in Lemma 54.1 gives that $\tilde{F}(s,1) = \tilde{g}(s)$.

So both $\tilde{f}$ and $\tilde{g}$ end at $\tilde{F}(\{1\}\times I) = \{e_1\}$, therefore $\tilde{F}$ is a path homotopy between $\tilde{f}$ and $\tilde{g}$. □

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# Theorem 54.4

**Theorem 54.4.** Let $p : E \to B$ be a covering map, and let $p(e_0) = b_0$. If $E$ is path connected, then the lifting correspondence

$$\phi : \pi(B, b_0) \to p^{-1}(b_0) \tag{5}$$

is surjective (onto). If $E$ is simply connected, it is bijective.

**Proof.** If $E$ is path connected, then, given any $e_1 \in p^{-1}(b_0)$, there is a path $\tilde{f}$ in $E$ from $e_0$ to $e_1$. Define $f = p \circ \tilde{f}$ and then $f(0) = p \circ \tilde{f}(0) = p(e_0) = b_0$ and $f(1) = p \circ \tilde{f}(1) = p(e_1) = b_0$ since $e_1 \in p^{-1}(b_0)$.

So $f$ is a loop in $B$ at $b_0$ and so

$$\phi([f]) = \tilde{f}(1) = e_1, \text{ so } \phi \text{ is onto } p^{-1}(b_0). \tag{6}$$

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# Theorem 54.4 Continued

Suppose $E$ is simply connected. Let $[f]$ and $[g]$ be two elements of $\pi_1(B, b_0)$ such that $\phi([f]) = \phi([f]) = \phi([g])$. Let $\tilde{f}$ and $\tilde{g}$ be the liftings of $f$ and $g$ respectively to paths in $E$ that begin at $e_0$. Since $\phi([f]) = \phi([g])$, the endpoints of $f$ and $g$ are the same, so $f(1) = g(1)$ and hence $\tilde{f}(1) = \tilde{g}(1)$. So $\tilde{f}$ and $\tilde{g}$ have the same initial and final points.

Since $E$ is simply connected, there is a path homotopy $\tilde{F}$ in $E$ between $\tilde{f}$ and $\tilde{g}$.

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Since $E$ is simply connected, there is a path homotopy $\tilde{F}$ in $E$ between $\tilde{f}$ and $\tilde{g}$. Since $\tilde{F}(I \times \{0\}) = \tilde{f}$ and $\tilde{F}(I \times \{1\}) = \tilde{g}$ then

$$\begin{aligned} p \circ \tilde{F}(I \times \{0\}) &= p \circ \tilde{f} = f \text{ and} \\ p \circ \tilde{F}(I \times \{1\}) &= p_0 \circ \tilde{g} = g \end{aligned} \tag{7}$$

since $\tilde{f}$ and $\tilde{g}$ are liftings of $f$ and $g$, respectively.

# Theorem 54.4 Continued

Suppose $E$ is simply connected. Let $[f]$ and $[g]$ be two elements of $\pi_1(B, b_0)$ such that $\phi([f]) = \phi([f]) = \phi([g])$. Let $\tilde{f}$ and $\tilde{g}$ be the liftings of $f$ and $g$ respectively to paths in $E$ that begin at $e_0$. Since $\phi([f]) = \phi([g])$, the endpoints of $f$ and $g$ are the same, so $f(1) = g(1)$ and hence $\tilde{f}(1) = \tilde{g}(1)$. So $\tilde{f}$ and $\tilde{g}$ have the same initial and final points.

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# Theorem 54.4 Continued

Since $p$ is continuous then $p \circ \tilde{F}$ is continuous and hence $p \circ \tilde{F}$ is a homotopy of $f$ with $g$. Therefore, $[f] = [g]$ and $\phi$ is one to one.

Since simply connected spaces are path connected [REF], then the first paragraph gives that $\phi$ is onto, so $\phi$ is bijective on a simply connected space. □

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# Theorem 54.5

**Theorem 54.5.** The fundamental group of $S^1$ is isomorphic to the additive group of integers, $\mathbb{Z}$.
**Proof.** Let $p : \mathbb{R} \to S^1$ be the covering map of Theorem 53.1:

$$p(s) = (cos(2\pi s), sin(2\pi s)) \quad (8)$$

Let $e_0 = 0$ and $b_0 = p(e_0) = (1, 0)$. Then $p^{-1}(b_0) = \mathbb{Z}$. Since $\mathbb{R}$ is simply connected, the lifting correspondence $\phi : \pi_1(s', b_0) \to \mathbb{Z}$ is bijective by Theorem 54.4. We now show that $\phi$ is a group homomorphism.

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Given $[f]$ and $[g]$ in $\pi_1(B, b_0)$, let $\tilde{f}$ and $\tilde{g}$ be liftings of $f$ and $g$, respectively, to paths in $\mathbb{R}$ beginning at 0. We know by the definition of $\phi$ that $\phi([f]), \phi([g]) \in \mathbb{Z}$ and are the endpoints in $\mathbb{R}$ of $\tilde{f}(1)$ and $\tilde{g}(1)$, respectively.

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Given $[f]$ and $[g]$ in $\pi_1(B, b_0)$, let $\tilde{f}$ and $\tilde{g}$ be liftings of $f$ and $g$, respectively, to paths in $\mathbb{R}$ beginning at 0. We know by the definition of $\phi$ that $\phi([f]), \phi([g]) \in \mathbb{Z}$ and are the endpoints in $\mathbb{R}$ of $\tilde{f}(1)$ and $\tilde{g}(1)$, respectively. Say $n = \phi([f])$ and $m = \phi([g])$. Let $\tilde{g}'$ be the path $\tilde{g}'(s) = n + \tilde{g}(s)$ in $\mathbb{R}$ (so $\tilde{g}'$ begins at $n$ and ends at $n + \tilde{g}(1) = n + m$).

# Theorem 54.5 Continued

Because $p(n+x) = p(x)$ ($p$ is a periodic function with period 1) for all $x \in \mathbb{R}$, the path $\tilde{g}'$ is a lifting of $g$ which begins at $n$, since $p \circ \tilde{g}' = g$ on $I$. Since $\tilde{f}$ ends at $n$, then $\tilde{f} * \tilde{g}'$ is defined and it is a lifting of $f * g$ beginning at 0, since

$$p(\tilde{f} * \tilde{g}') = p(\tilde{f}) * p(\tilde{g}') = f * g \quad (9)$$

The end point of $f * g$ is $n + m$ and the end point of $\tilde{g}'$ is $n + m$ (we need to introduce $\tilde{g}'$ because $\tilde{f} * \tilde{g}$ is not defined — $\tilde{f}$ ends at $n$ and $\tilde{g}$ begins at 0).

# Theorem 54.5 Continued

Because $p(n+x) = p(x)$ ($p$ is a periodic function with period 1) for all $x \in \mathbb{R}$, the path $\tilde{g}'$ is a lifting of $g$ which begins at $n$, since $p \circ \tilde{g}' = g$ on $I$. Since $\tilde{f}$ ends at $n$, then $\tilde{f} * \tilde{g}'$ is defined and it is a lifting of $f * g$ beginning at 0, since

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The end point of $f * g$ is $n + m$ and the end point of $\tilde{g}'$ is $n + m$ (we need to introduce $\tilde{g}'$ because $\tilde{f} * \tilde{g}$ is not defined — $\tilde{f}$ ends at $n$ and $\tilde{g}$ begins at 0). So

$$\phi([f * g]) = \phi([f] * [g]) = m + n \tag{10}$$

since $\tilde{f} * \tilde{g}'$ ends at $m + n$ and

$$\phi([f]) + \phi([g]) = n + m \tag{11}$$

Since $\tilde{f}$ ends at $n$ and $\tilde{g}$ ends at m.

# Theorem 54.5 Continued

Because $p(n+x) = p(x)$ ($p$ is a periodic function with period 1) for all $x \in \mathbb{R}$, the path $\tilde{g}'$ is a lifting of $g$ which begins at $n$, since $p \circ \tilde{g}' = g$ on $I$. Since $\tilde{f}$ ends at $n$, then $\tilde{f} * \tilde{g}'$ is defined and it is a lifting of $f * g$ beginning at 0, since

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Since $\tilde{f}$ ends at $n$ and $\tilde{g}$ ends at m.

# Theorem 54.5 Continued

Therefore,

$$\phi([f] * [g]) = \phi([f]) + \phi([g]) \tag{12}$$

and $\phi$ is a group homomorphism.

Now if $\phi : G \to G'$ is a group homomorphism, then the set $\phi(G)$ is a subgroup of $G'$ (see Fraleigh, Theorem 13.12(3)). Every subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ (see Fraleigh's Corollary 6.7 and Theorem 11.12) so $\phi$ is an group isomorphism between $\pi_1(B, b_0)$ and $\mathbb{Z}$. □

# Theorem 54.5 Continued

Therefore,

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Now if $\phi : G \to G'$ is a group homomorphism, then the set $\phi(G)$ is a subgroup of $G'$ (see Fraleigh, Theorem 13.12(3)). Every subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ (see Fraleigh's Corollary 6.7 and Theorem 11.12) so $\phi$ is an group isomorphism between $\pi_1(B, b_0)$ and $\mathbb{Z}$. □