Introduction to Topology

Chapter 9. The Fundamental Group

Section 55. Retraction and Fixed Points—Proofs of Theorems

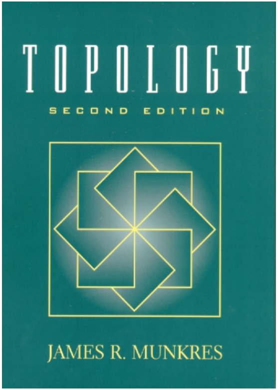

Table of contents

Lemma 55.1

Lemma 55.1. If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion $j : A \to X$ is injective (one to one)

Proof. Notice that inclusion $j : a \to X$ is just the identity map $j(a) = a$ for all $a \in A$ (if you like, it is the identity on X restricted to A). If $v : X \to A$ is a retraction, then $v \circ j : A \to A$ is the identity map of A.

By Theorem 52.4, the induced homomorphism satisfies $(v \circ j)_* = v_* \circ j_*$ and since $v \circ j$ is the identity map, then $v_* \circ j_*$ is the identity homomorphism on $\pi_1(A, a)$ (where 'a' is some "base point" in A).

Lemma 55.1

Lemma 55.1. If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion $j : A \to X$ is injective (one to one)

Proof. Notice that inclusion $j : a \to X$ is just the identity map $j(a) = a$ for all $a \in A$ (if you like, it is the identity on X restricted to A). If $v : X \to A$ is a retraction, then $v \circ j : A \to A$ is the identity map of A.

By Theorem 52.4, the induced homomorphism satisfies $(v \circ j)_* = v_* \circ j_*$ and since $v \circ j$ is the identity map, then $v_* \circ j_*$ is the identity homomorphism on $\pi_1(A, a)$ (where 'a' is some "base point" in A).

So v_* is a left inverse of j_* and hence j_* is one to one. □

Lemma 55.1

Lemma 55.1. If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion $j : A \to X$ is injective (one to one)

Proof. Notice that inclusion $j : a \to X$ is just the identity map $j(a) = a$ for all $a \in A$ (if you like, it is the identity on X restricted to A). If $v : X \to A$ is a retraction, then $v \circ j : A \to A$ is the identity map of A.

By Theorem 52.4, the induced homomorphism satisfies $(v \circ j)_* = v_* \circ j_*$ and since $v \circ j$ is the identity map, then $v_* \circ j_*$ is the identity homomorphism on $\pi_1(A, a)$ (where 'a' is some "base point" in A).

So v_* is a left inverse of j_* and hence j_* is one to one. □

Theorem 55.2 No-Retraction Theorem

Theorem 55.2. There is no retraction of the closed disk B^2 to the circle S^1.

Proof. If S^1 were a retract of B^2, then the homomorphism $j_* : \pi_1(S^1, a) \to \pi_1(B^2, b)$ induced by inclusion $j : S^1 \to B^2$ would be one to one by Lemma 55.1. But $\pi_1(S^1, a) \cong \mathbb{Z}$ and, since B^2 is simply connected, $\pi_1(B^2, b) = \{e\}$.

So j_* cannot be one to one. Hence, there is no such retract. $\square$

Theorem 55.2 No-Retraction Theorem

Theorem 55.2. There is no retraction of the closed disk B^2 to the circle S^1.

Proof. If S^1 were a retract of B^2, then the homomorphism $j_* : \pi_1(S^1, a) \to \pi_1(B^2, b)$ induced by inclusion $j : S^1 \to B^2$ would be one to one by Lemma 55.1. But $\pi_1(S^1, a) \cong \mathbb{Z}$ and, since B^2 is simply connected, $\pi_1(B^2, b) = \{e\}$.

So j_* cannot be one to one. Hence, there is no such retract. $\square$

Corollary 55.4

Corollary 55.4. The inclusion map $j : S^1 \to \mathbb{R}\backslash\{(0,0)\}$ is not nulhomotopic. The identity map $\iota : S^1 \to S^1$ is not nulhomotopic.

Proof. A retraction of $\mathbb{R}^2\backslash\{(0,0)\}$ onto S^1 is given at the beginning of this section. So, by Lemma 55.1, the induced homomorphism $j_* : \pi(S^1, a) \to \pi_1(\mathbb{R}^2\backslash\{(0,0)\}, b)$ is one to one.

Since $\pi_1(S^1, a) \cong \mathbb{Z}$ then j_* does not map everything to the identity of $\pi_1(\mathbb{R}^2\backslash\{(0,0)\}, b)$ — that is, j_* is a nontrivial map. Therefore by Theorem 55.3 (the contrapositive of (1) $\Rightarrow$ (3)), j is not nulhomotopic.

Corollary 55.4

Corollary 55.4. The inclusion map $j : S^1 \to \mathbb{R}\backslash\{(0,0)\}$ is not nulhomotopic. The identity map $\iota : S^1 \to S^1$ is not nulhomotopic.

Proof. A retraction of $\mathbb{R}^2\backslash\{(0,0)\}$ onto S^1 is given at the beginning of this section. So, by Lemma 55.1, the induced homomorphism $j_* : \pi(S^1, a) \to \pi_1(\mathbb{R}^2\backslash\{(0,0)\}, b)$ is one to one.

Since $\pi_1(S^1, a) \cong \mathbb{Z}$ then j_* does not map everything to the identity of $\pi_1(\mathbb{R}^2\backslash\{(0,0)\}, b)$ — that is, j_* is a nontrivial map. Therefore by Theorem 55.3 (the contrapositive of (1) $\Rightarrow$ (3)), j is not nulhomotopic.

Since ι is the identity map, then the induced homomorphism ι_* is the identity by Theorem 52.4. So $\iota_* : \mathbb{Z} \to \mathbb{Z}$ is one to one and nontrivial. Again by Theorem 55.3, ι is not nulhomotopic. $\square$

Corollary 55.4

Corollary 55.4. The inclusion map $j : S^1 \to \mathbb{R}\backslash\{(0,0)\}$ is not nulhomotopic. The identity map $\iota : S^1 \to S^1$ is not nulhomotopic.
Proof. A retraction of $\mathbb{R}^2\backslash\{(0,0)\}$ onto S^1 is given at the beginning of this section. So, by Lemma 55.1, the induced homomorphism $j_* : \pi(S^1, a) \to \pi_1(\mathbb{R}^2\backslash\{(0,0)\}, b)$ is one to one.

Since $\pi_1(S^1, a) \cong \mathbb{Z}$ then j_* does not map everything to the identity of $\pi_1(\mathbb{R}^2\backslash\{(0,0)\}, b)$ — that is, j_* is a nontrivial map. Therefore by Theorem 55.3 (the contrapositive of (1) $\Rightarrow$ (3)), j is not nulhomotopic.

Since ι is the identity map, then the induced homomorphism ι_* is the identity by Theorem 52.4. So $\iota_* : \mathbb{Z} \to \mathbb{Z}$ is one to one and nontrivial. Again by Theorem 55.3, ι is not nulhomotopic. □

Theorem 55.5

Theorem 55.5. Given a nonvanishing vector field on B^2, there exists a point of S^1 where the vector field points directly inward and a point of S^1 where the vector field points directly outward.

Proof. Let $(\vec{x}, v\vec{x})$ be a nonvanishing vector field on B^2. ASSUME that $v(\vec{x})$ does not point directly inward at any point $\vec{x}$ of S^1. Notice that $v : B^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$ (since v is nonvanishing).

Let w bw the restriction of v to S^1. Because w extends from S^1 to a map of B^2 into $\mathbb{R}^2 \setminus \{(0,0)\}$ then by Theorem 55.2 ((2) $\Rightarrow$ (1)), w is nulhomotopic.

Theorem 55.5

Theorem 55.5. Given a nonvanishing vector field on B^2, there exists a point of S^1 where the vector field points directly inward and a point of S^1 where the vector field points directly outward.

Proof. Let $(\vec{x}, v\vec{x})$ be a nonvanishing vector field on B^2. ASSUME that $v(\vec{x})$ does not point directly inward at any point $\vec{x}$ of S^1. Notice that $v : B^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$ (since v is nonvanishing).

Let w bw the restriction of v to S^1. Because w extends from S^1 to a map of B^2 into $\mathbb{R}^2 \setminus \{(0,0)\}$ then by Theorem 55.2 ((2) $\Rightarrow$ (1)), w is nulhomotopic.

Theorem 55.5 Continued

Consider the straight line mapping of w to S^1:

$$F(\vec{x}, t) = t\vec{x} + (1 - t)w(\vec{x}) \textit{ for } \vec{x} \in S^1 \tag{1}$$

(We next show this is a path homotopy in $\mathbb{R}^2 \backslash \{(0,0)\}$). Notice that $F(\vec{x}, 0) = w(\vec{x})$ and $F(\vec{x}, 1) = \vec{x}$. Now neither S^1 nor $w|_{S^1}$ contains $(0,0)$. So $F(\vec{x}, t) \neq (0,0)$ for $t = 0, 1$.

If $F(\vec{x}, t) = (0,0)$ for some t with $0 < t < 1$, then

$$t\vec{x} + (1 - t)w(\vec{x}) = (0,0) \textit{ so that } w(\vec{x}) = \frac{-t}{1-t}\vec{x} \tag{2}$$

Theorem 55.5 Continued

Consider the straight line mapping of w to S^1:

$$F(\vec{x}, t) = t\vec{x} + (1-t)w(\vec{x}) \, for \, \vec{x} \in S^1 \tag{1}$$

(We next show this is a path homotopy in $\mathbb{R}^2 \setminus \{(0,0)\}$). Notice that $F(\vec{x}, 0) = w(\vec{x})$ and $F(\vec{x}, 1) = \vec{x}$. Now neither S^1 nor $w|_{S^1}$ contains $(0,0)$. So $F(\vec{x}, t) \neq (0,0)$ for $t = 0, 1$.

If $F(\vec{x}, t) = (0,0)$ for some t with $0 < t < 1$, then

$$t\vec{x} + (1-t)w(\vec{x}) = (0,0) \, so \, that \, w(\vec{x}) = \frac{-t}{1-t}\vec{x} \tag{2}$$

that is, $w(\vec{x})$ is a negative scalar multiple of $\vec{x} \in S^1$. But this means that $w(\vec{x})$ points directly inward at $\vec{x}$.

Theorem 55.5 Continued

Consider the straight line mapping of w to S^1:

$$F(\vec{x}, t) = t\vec{x} + (1 - t)w(\vec{x}) \, for \, \vec{x} \in S^1 \tag{1}$$

(We next show this is a path homotopy in $\mathbb{R}^2 \setminus \{(0,0)\}$). Notice that $F(\vec{x}, 0) = w(\vec{x})$ and $F(\vec{x}, 1) = \vec{x}$. Now neither S^1 nor $w|_{S^1}$ contains $(0,0)$. So $F(\vec{x}, t) \neq (0,0)$ for $t = 0, 1$.

If $F(\vec{x}, t) = (0,0)$ for some t with $0 < t < 1$, then

$$t\vec{x} + (1 - t)w(\vec{x}) = (0,0) \, so \, that \, w(\vec{x}) = \frac{-t}{1 - t}\vec{x} \tag{2}$$

that is, $w(\vec{x})$ is a negative scalar multiple of $\vec{x} \in S^1$. But this means that $w(\vec{x})$ points directly inward at $\vec{x}$.

Theorem 55.5 Continued

So it must be that $F(\vec{x}, t) \neq (0,0)$ for all $t \in [0,1]$ and so

$$F : S^1 \times I \to \mathbb{R}^2 - \{(0,0)\} \tag{3}$$

So F is a path homotopy between the path w and the inclusion map $j : S^1 \to \mathbb{R}^2 \backslash \{(0,0)\}$ (j is a path in $\mathbb{R}^2 - \{(0,0)\}$ since its image is simply $\{(x,y) | x^2 + y^2 = 1\}$) IN $\mathbb{R}^2 - \{(0,0)\}$. But inclusion map j is not nulhomotopic by Corollary 55.4. Since w and j are homotopic, then w is not nulhomotopic.

Theorem 55.5 Continued

So it must be that $F(\vec{x}, t) \neq (0,0)$ for all $t \in [0,1]$ and so

$$F : S^1 \times I \to \mathbb{R}^2 - \{(0,0)\} \tag{3}$$

So F is a path homotopy between the path w and the inclusion map $j : S^1 \to \mathbb{R}^2 \backslash \{(0,0)\}$ (j is a path in $\mathbb{R}^2 - \{(0,0)\}$ since its image is simply $\{(x,y) | x^2 + y^2 = 1\}$) IN $\mathbb{R}^2 - \{(0,0)\}$. But inclusion map j is not nulhomotopic by Corollary 55.4. Since w and j are homotopic, then w is not nulhomotopic.

Theorem 55.5 Continued

So we have shown that w is both nulhomotopic and not nulhomotopic. This contradiction shows that the assumption that $v(\vec{x})$ does not point directly inward at any point of S^1 is false.

To show that v points directly outward at some point of S^1, we apply the above result to the nonvanishing vector field $(\vec{x}, -v(\vec{x}))$. $\square$

Theorem 55.5 Continued

So we have shown that w is both nulhomotopic and not nulhomotopic. This contradiction shows that the assumption that $v(\vec{x})$ does not point directly inward at any point of S^1 is false.

To show that v points directly outward at some point of S^1, we apply the above result to the nonvanishing vector field $(\vec{x}, -v(\vec{x}))$. $\square$

Theorem 55.6 The Brouwer Fixed-Point Theorem for the Disk

Theorem 55.6. If $f : B^2 \to B^2$ is continuous, then there exists a point $\vec{x} \in B^2$ such that $f(\vec{x}) = \vec{x}$

Proof. ASSUME to th contrary that $f(\vec{x}) \neq \vec{x}$ for every $\vec{x} \in B^2$. Then define $v(\vec{x}) = f(\vec{x}) - \vec{x}$. So $(\vec{x}, v(\vec{x}))$ is a nonvanishing vector field on B^2.

Then the vector field must point directly outward at some point $\vec{x}$ of S^1 by Theorem 55.5, say $v(\vec{x}) = a\vec{x}$ where $a > 0$. Then $v(\vec{x}) = f(\vec{x}) - \vec{x} = ax$ or $f(\vec{x}) = (1 + a)\vec{x}$.

Theorem 55.6 The Brouwer Fixed-Point Theorem for the Disk

Theorem 55.6. If $f : B^2 \to B^2$ is continuous, then there exists a point $\vec{x} \in B^2$ such that $f(\vec{x}) = \vec{x}$

Proof. ASSUME to th contrary that $f(\vec{x}) \neq \vec{x}$ for every $\vec{x} \in B^2$. Then define $v(\vec{x}) = f(\vec{x}) - \vec{x}$. So $(\vec{x}, v(\vec{x}))$ is a nonvanishing vector field on B^2.

Then the vector field must point directly outward at some point $\vec{x}$ of S^1 by Theorem 55.5, say $v(\vec{x}) = a\vec{x}$ where $a > 0$. Then $v(\vec{x}) = f(\vec{x}) - \vec{x} = ax$ or $f(\vec{x}) = (1 + a)\vec{x}$.

But then $f(\vec{x}) \notin B^2$, a contradiction. This contradiction shows that the assumption is false, and so $f(\vec{x}) = \vec{x}$ for some $\vec{x} \in B^2$. □

Theorem 55.6 The Brouwer Fixed-Point Theorem for the Disk

Theorem 55.6. If $f : B^2 \to B^2$ is continuous, then there exists a point $\vec{x} \in B^2$ such that $f(\vec{x}) = \vec{x}$

Proof. ASSUME to th contrary that $f(\vec{x}) \neq \vec{x}$ for every $\vec{x} \in B^2$. Then define $v(\vec{x}) = f(\vec{x}) - \vec{x}$. So $(\vec{x}, v(\vec{x}))$ is a nonvanishing vector field on B^2.

Then the vector field must point directly outward at some point $\vec{x}$ of S^1 by Theorem 55.5, say $v(\vec{x}) = a\vec{x}$ where $a > 0$. Then $v(\vec{x}) = f(\vec{x}) - \vec{x} = ax$ or $f(\vec{x}) = (1 + a)\vec{x}$.

But then $f(\vec{x}) \notin B^2$, a contradiction. This contradiction shows that the assumption is false, and so $f(\vec{x}) = \vec{x}$ for some $\vec{x} \in B^2$. □