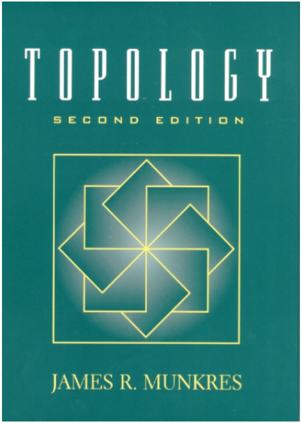
Introduction to Topology

Chapter 9. The Fundamental Group

Section 57. The Borsuk-Ulam Theorem—Proofs of Theorems



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Theorem 57.1. If $h : S^1 \to S^1$ is continuous and antipode-preserving, then h is not nulhomotopic.

Proof. Let b_0 be the point $(1, 0) \in S^1$. Let $p : S^1 \to S^1$ be a rotation of S^1 that maps $h(b_0)$ to b_0. Since p preserves antipodes (it is a "rigid" rotation — antipodal pairs remain antipodal pairs under h), then $p \circ h$ preserves antipodes.

Furthermore, if H were a homotopy between h (since $h : S^1 \to S^1$ is continuous, it is a path in S^1) and a constant map (i.e., if h were nulhomotopic) then $p \circ H$ would be a homotopy between $p \circ h$ and a constant map. So, without loss of generality, we assume $h(b_0) = b_0$, otherwise we consider $p \circ h$ as opposed to h alone.

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Theorem 57.1 Continued

STEP 1 Let $q : S^1 \to S^1$ be the map $q(z) = z^2$ where $z \in \mathbb{C}$ (so now $S^1 = \{z \in \mathbb{C} | \, |z| = 1\}$). In real coordinates, we have for $(cos\theta, sin\theta) \in S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ that $q(cos\theta, sin\theta) = (cos(2\theta), sin(2\theta))$. The map q maps open sets to open sets (and equivalently closed sets to closed sets); that is, q is a "closed map" (equivalently, q is an open map), q is continuous, and q is surjective (onto). Such a map (closed, continuous, surjective) is called a quotient map (see Section 22).

Theorem 57.1 Continued

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Notice that for the distinct points $(cos\theta_1, sin\theta_1), (cos(\theta_1 + \pi), sin(\theta_1 + \pi)) \in S^1$ we have $q(cos\theta_1, sin\theta_1) = (cos(2\theta_1), sin(2\theta_1)) = (cos(2\theta_1 + 2\pi), sin(2\theta_1 + 2\pi)) = q(cos(\theta_1 + \pi), sin(\theta_1 + \pi))$. So the inverse image under q of any point of S^1 consists of two antipodal points z and $-z$ of S^1.

Theorem 57.1 Continued

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Theorem 57.1 Continued

Because $h(-z) = -h(z)$ by hypothesis, we have that $q(h(-z)) = q(-h(z)) = q(h(z))$. So for any point, $s \in S^1$, $q^{-1}(\{s\}) = \{z, -z\}$, say. Then $q \circ h$ is constant on $q^{-1}(\{s\})$ for all $s \in S^1$.

Therefore by Theorem 22.2 (with $p = q$ and $g = q \circ h$) the map $q \circ h$ induces a continuous map $k : S^1 \to S^1$ (this is map f in Theorem 22.2) such that $k \circ q = q \circ h$ ($g = f \circ p$ in Thm 22.2)

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Note that $q(b_0) = h(b_0) = b_0$ (from above) so that $k(b_0) = b_0$ as well. Also, $h(-b_0) = -b_0$ since h is antipode preserving by hypothesis.

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Theorem 57.1 Continued

STEP 2 We now show that the induced homomorphism $k_* : \pi_1(S^1, b_0) \to \pi(S^1, b_0)$ is nontrivial.
We first show that $q : S^1 \to S^1$, $q(z) = z^2$, is a covering map. The argument is similar to that as given in the proof of Theorem 53.1.

Consider U_1, the open subset of S^1 consisting of all points lying in the upper half plane. Then the "argument" of the points in U_1 are between 0 and π.

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Then $p^{-1}(U_1)$ consists of the poitns in S^1 in the first quadrant whose square is in U_1 (these points have arguments between 0 and $\frac{\pi}{2}$), AND the points in S^1 in the third quadrant whose square is in U_1 (these points have arguments between π and $\frac{3\pi}{2}$, so the squares of these points are between 2π and 3π, or equivalently between 0 and π).

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Theorem 57.1 Continued

So q carries each of these two sets homeomorphically to U_1. So U_1 is evenly covered by q.

Similarly, the four pieces of S^1 determined y intersecting it with the open lower, left, and right half planes are evenly covered by q. So q is a covering map.

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Second, let $\tilde{f}$ be a path in S^1 from b_0 to $-b_0$ (so this path consists of points in S^1 with arguments ranging over an interval of length π). So the loop $f = q \circ \tilde{f}$ consists of points in S^1 with arguments ranging over an interval of length 2π and so can be considered a loop from b_0 to b_0.

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Now $\tilde{f}$ is a lifting of f in S^1 that begins at b_0 but does not end at b_0.

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Theorem 57.1 Continued

Now if $\tilde{q}$ is a lifting of a loop starting at b_0 and path homotopic to a constant (i.e., nulhomotopic) then $\tilde{q}$ begins at b_0 and ends at b_0*. So, by Thereom 54.3, since q is a covering map by above, it must be that f and q are not path homotopic and so $[f]$ is a nontrivial (i.e., non-identity) element of $\pi_1(S^1, b_0)$.

Now we show that k_* is nontrivial. As above, let $\tilde{f}$ be a path in S^1 from b_0 to $-b_0$ and let f be the loop $q \circ \tilde{f}$.

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$$\begin{aligned} k_*[f] = [k \circ f] = [k \circ (q \circ \tilde{f})] &= [(k \circ q) \circ \tilde{f}] \\ &= [(q \circ h) \circ \tilde{f}] \text{ since } k \circ q = q \circ h \text{ by STEP 1} \\ &= [q \circ (h \circ \tilde{f})] \end{aligned} \tag{1}$$

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Since h is antipode preserving (by hypothesis), then $h \circ \tilde{f}$ is a path in S^1 from b_0 to $-b_0$.

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Since h is antipode preserving (by hypothesis), then $h \circ \tilde{f}$ is a path in S^1 from b_0 to $-b_0$.

Theorem 57.1 Continued

So by the previous paragraph (with $\tilde{f}$ of that paragraph replace with $h \circ \tilde{f}$ of this paragraph), $[q \circ (h \circ \tilde{f}]$ (this is $f = q \circ \tilde{f}$ of the previous paragraph) is nontrivial. So k_* is a nontrivial homomorphism from $\pi_1(S^1, b_0)$ to $\pi_1(S^1, b_0)$.

Theorem 57.1 Continued

STEP 3 The homomorphism induced by h, $h_* : \pi_1(S^1, b_0) \to \pi_1(S^1, b_0)$ is one to one (injective) since it is a homomorphism from an infinite cyclic group to an infinite cyclic group (recall $\pi_1(S^1, b_0) \cong \mathbb{Z}$ by Theorem 54.5, the image of $\pi_1(S^1, b_0)$ is a subgroup of $\pi_1(S^1, b_0) \cong \mathbb{Z}$ and a subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some $n \in \mathbb{N}$).

Now consider the homomorphism induced by q, $q_* : \pi_1(S^1, b_0) \to \pi_1(S^1, b_0)$. FOr a loop wrapping around S^1 n times, q maps this to a loop mapping around S^1 $2n$ times.

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So h_* is not the trivial homomorphism and so by Lemma 55.3 (NOT (3) $\Rightarrow$ NOT (1)) h is not nulhomotopic. □

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Theorem 57.2

Theorem 57.2. There is no continuous antipode-preserving map $g : S^2 \to S^1$.

Proof. ASSUME $g : S^2 \to S^1$ is continuous and antipode-preserving. Interpret S^1 as "the equation" of S^2. Then $g|_{S^1}$ is a continuous antipode-preserving map of S^1 to itself, denote $h = g|_{S^1}$.

By Theorem 57.1, h is not nulhomotopic. But the upper (closed) hemisphere is homeomorphic to the dish B^2 (just squash the hemisphere down with a projection). So g is a continuous extension of h from S^1 to B^2.

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Then by Lemma 55.3 ((2) $\Rightarrow$ (1)) h is nulhomotopic, a CONTRADICTION. So the assumption that continuous $g : S^2 \to S^1$ exists is false and the result follows. $\square$

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Theorem 57.3 Borsuk-Ulam Theorem for S^2

Theorem 57.3. Given a continuous map $f : S^2 \to \mathbb{R}^2$, there is a point $x \in S^2$ such that $f(x) = f(-x)$.
Proof. ASSUME that $f(x) \neq f(-x)$ for all $x \in S^2$. Then,

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|} \tag{2}$$

is continuous since f is continuous and $g : S^2 \to S^1$ since each value $g(x)$ is of norm 1. Also, $g(x) = g(-x)$ for all $x \in S^2$ and so g is antipode-preserving.

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Theorem 57.4 The Bisection Theorem

Theorem 57.4. Given two bounded polygonal regions in $\mathbb{R}^2$, there exists a line in $\mathbb{R}^2$ that bisects each of them.

Proof. We embed the problem in $\mathbb{R}^3$ and consider two polygonal regions A_1 and A_2 in the plane $\mathbb{R}^2 \times \{1\}$ in $\mathbb{R}^3$.

Given a point $\vec{x} \in S^2$, consider the plane P in $\mathbb{R}^3$ passing through the origin that has $\vec{u}$ as a unit normal vector. For $i = 1, 2$ let $f_i(u)$ be the area of the portion of A_i that lies on the same side of P as does vector $\vec{u}$ when in standard position.

Theorem 57.4 The Bisection Theorem

Theorem 57.4. Given two bounded polygonal regions in $\mathbb{R}^2$, there exists a line in $\mathbb{R}^2$ that bisects each of them.

Proof. We embed the problem in $\mathbb{R}^3$ and consider two polygonal regions A_1 and A_2 in the plane $\mathbb{R}^2 \times \{1\}$ in $\mathbb{R}^3$.

Given a point $\vec{x} \in S^2$, consider the plane P in $\mathbb{R}^3$ passing through the origin that has $\vec{u}$ as a unit normal vector. For $i = 1, 2$ let $f_i(u)$ be the area of the portion of A_i that lies on the same side of P as does vector $\vec{u}$ when in standard position.

If $\vec{u}$ is the unit vector $\hat{k}$ then define $f_i(\vec{u}) =$ area of A_i. If $\vec{u}$ is the unit vector $-\hat{k}$ then define $f_i(\hat{u}) = 0$.

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Replacing $\vec{u}$ by $-\vec{u}$ gives the same plane P, but the other half-space so that $f_i(\vec{u}) + f_i(-\vec{u}) =$ area of A_i.

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Theorem 57.4 The Bisection Theorem Continued

Now consider the map $F : S62 \to \mathbb{R}^2$ given by $F(\vec{u}) = (f_i(\vec{u}), f_2(\vec{u}))$. Then F is continuous (WE CLAIM) and so by the Borsuk-Ulam Theorem (Thm 57.3) there is a point $\vec{u} \in S^2$ for which $F(\vec{u}) = F(-\vec{u})$. Then $f_i(\vec{u}) = f_i(-\vec{u})$ for $i = 1, 2$. That is,

$$f_i(\vec{u}) + f_i(-\vec{u}) = 2f_i(\vec{u}) = \text{ } \mathit{area\ of}\ A_i \tag{3}$$

and $f_i(\vec{u}) = \frac{1}{2}(\text{area of } A_i)$. □

Theorem 57.4 The Bisection Theorem Continued

Now consider the map $F : S62 \to \mathbb{R}^2$ given by $F(\vec{u}) = (f_i(\vec{u}), f_2(\vec{u}))$. Then F is continuous (WE CLAIM) and so by the Borsuk-Ulam Theorem (Thm 57.3) there is a point $\vec{u} \in S^2$ for which $F(\vec{u}) = F(-\vec{u})$. Then $f_i(\vec{u}) = f_i(-\vec{u})$ for $i = 1, 2$. That is,

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