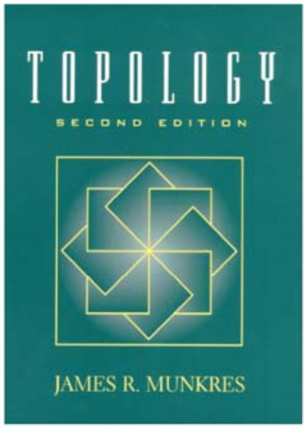
Introduction to Topology

Chapter 9. The Fundamental Group

Section 58. Deformation Retracts and Homotopy Type—Proofs of Theorems



Lemma 58.1

Lemma 58.1. Let $h, k : (X, x_0) \to (Y, y_0)$ be continuous maps. If h and k are homotopic and if the image of the base point $x_0 \in X$ remains fixed at y_0 during the homotopy then the induced homomorphisms h_* and k_* are equal.

Proof. Let $H : X \times I \to Y$ be a homotopy between h and k such that $H(x_0, t) = y_0$ for all $t \in I$. Let f be a loop in X based at x_0. Consider the composition

$$I \times I \xrightarrow{\text{f} \times \text{ID}} X \times I \xrightarrow{\text{H}} Y \tag{1}$$

Now $H(x, 0) = h(x)$ and $H(x, 1) = k(x)$ by the definition of homotopy. Also, $(f \times ID)(x, 0) = (f(x), 0)$ and $(f \times ID)(x, 1) = (f(x), 1)$ so $H \circ (f \times ID)(x, 0) = h(f(x))$ and $H \circ (f \times ID)(x, 1) = k(f(x))$. So $H \circ (f \times ID)$ is a homotopy from $h(f(x)) = h \circ f$ and $k(f(x)) = k \circ f$.

Lemma 58.1 Continued

Since f is a loop based at x_0, then $h \circ f$ and $k \circ f$ are loops based at y_0 and since H maps $\{x_0\} \times I$ to y_0, the homotopy is a path homotopy. So,

$$\begin{aligned} h_*[f] &= [h \circ f] \\ &= [k \circ f] \textit{ because of the path homotopy} \\ &= k_*[f] \end{aligned} \tag{2}$$

for all loops f based at x_0. So $h_* = k_*$ on $\pi_1(X, x_0)$. □

Theorem 58.2

Theorem 58.2. The inclusion map $j : S^n \to \mathbb{R}^{n+1} \setminus \{\vec{0}\}$ induces an isomorphism of fundamental groups $\pi_1(S^n, x_0)$ and $\pi_1(\mathbb{R}^{n+1} \setminus \{\vec{0}\}, x_0)$ for $n \geq 1$.

Proof. Let $X = \mathbb{R}^{n+1} \setminus \{\vec{0}\}$ and let $b_0 = (1, 0, ..., 0)$. Define $r : X \to S^n$ as $r(\vec{x}) = \vec{x}/\|\vec{x}\|$. Then $r \circ j$ is the identity map on S^n and so $(r \circ j)_* = r_* \circ j_*$ is the identity homomorphism on $\pi_1(S^n, b_0)$.

Now consider $j \circ r : x \to x$:

$$X \xrightarrow{\text{r}} S^n \xrightarrow{\text{j}} X \tag{3}$$

Theorem 58.2 Continued

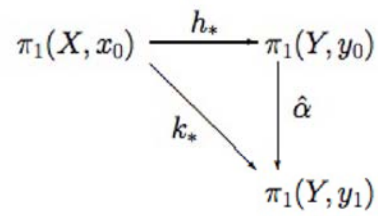
Now $j \circ r$ is obviously not the identity map on X. But $j \circ r$ is straight line homotopic to the identity on X since

$$H(\vec{x}, t) = (1-t)\vec{x} + t\vec{x}/\|\vec{x}\| \tag{4}$$

is a homotopy between the identity map on X (When $t = 0$) and $j \circ r$ (when $t = 1$).
Notice $H(x, t)$ is never $\vec{0}$ since $(1-t) + t\|\vec{x}\|$ is a number between 1 and $1/\|\vec{x}\|$. Also, $b_0 = (1, 0, \dots, 0)$ remains fixed during the homotopy since $\|b_0\| = 1$. By Lemma 58.1, the induced homomorphism $(j \circ r)_* = j_* \circ r_*$ is the identity homomorphism from $\pi_1(X, b_0)$ to itself. So j_* and r_* are invertible and hence one to one and onto (bijective). So j_* induces an isomorphism from $\pi_1(S^n, b_0)$ to $\pi_1(X, b_0)$. □

Lemma 58.4

Lemma 58.4. Let $h, k : X \to Y$ be continuous maps. Let $h(x_0) = y_0$ and $k(x_0) = y_1$. if h and k are homotopic, there is a path α in Y from y_0 to y_1 such that $k_* = \hat{\alpha} \circ h_*$ (where k_* and h_* are induced homomorphisms on the fundamental group and $\hat{\alpha} : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ is defined as $\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$—see Figure Below). Indeed, if $H : X \times I \to Y$ is the homotopy between h and k, then α is the path $\alpha(t) = H(x_0, t)$.



Lemma 58.4

Proof. Let f be an arbitrary loop in X based at x_0. We must show that

$$k_*([f]) = \hat{\alpha}(h_*([f])) \tag{5}$$

By definition of the induced homomorphism, their equation is equivalent to $[k \circ f] = \hat{\alpha}([h \circ f]) = [\bar{\alpha}] * [h \circ f] * [\alpha]$ by the definition of $\hat{\alpha}$, where $\bar{\alpha}$ is the reverse of α. Since $[\alpha] * [\bar{\alpha}]$ is the identity, the equation becomes

$$[\alpha] * [k \circ f] = [h \circ f] * [\alpha] \quad (*) \tag{6}$$

Consider the loops f_0 and f_1 in the space $X \times I$ given as

$$f_0(s) = (f(s), 0) \textit{ and } f_1(s) = (f(s), 1) \tag{7}$$

Consider the path c in $X \times I$ given by

$$c(t) = (x_0, t) \tag{8}$$

Lemma 58.4 Continued

Since H is a homotopy between h and k, then

$$\begin{aligned} (\dagger)\ \ H \circ f_0 &= H(f(s), 0) = h(f(s)) = h \circ f \textit{ and} \\ (\dagger\dagger)\ \ H \circ f_1 &= H(f(s), 1) = k(f(s)) = k \circ f \end{aligned} \tag{9}$$

While we define $\alpha(t)$ as $H(x_0, t) = H \circ c$. $(\dagger\dagger\dagger)$
Notice that $H(x_0, 0) = h(x_0) = y_0$ and $H(x_0, 1) = k(x_0) = y_1$. So in fact α is a path from y_0 to y_1.

Let $F : I \times I \to X \times I$ be the map $F(s, t) = (f(s), t)$. Consider the following paths in $I \times I$ which run along the four edges of $I \times I$:

$$\begin{aligned} \beta_0(s) = (s, 0) \quad \beta_1(s) = (s, 1) \\ \gamma_0(t) = (0, t) \quad \gamma_1(t) = (1, t) \end{aligned} \tag{10}$$

Then $F \circ \beta_0 = F(s, 0) = (f(s), 0)$ and $F \circ \beta_1 = F(s, 1) = (f(s), 1)$, while $F \circ \gamma_0 = F(0, t) = (f(0), t) = (x_0, t) = c(t)$ and $F \circ \gamma_1 = F(1, t) = (f(1), t) = (x_0, t) = c(t)$.

Lemma 58.4 Continued

Now paths $\beta_0 * \gamma_1$ and $\gamma_0 * \beta_1$ are paths in $I \times I$ (along the boundary) from $(0, 0)$ to $(1, 1)$. Since $I \times I$ is convex (See Example 51.1), there is a path homotopy G between them (say from $\beta_0 * \gamma_1$ to $\gamma_0 * \beta_1$).

Then for $F \circ G : (X \times I) \times I$ we have $(F \circ G)(\beta_0 * \gamma_1, 0) = F(\beta_0 * \gamma_1)$, $(F \circ G)(\beta_0 * \gamma_1, 0) = F(\beta_0 * \gamma_1) = F(\beta_0) * F(\gamma_1) = (f(s), 0) * c(t) = f_0 * c$, and
$(F \circ G)(\beta_0 * \gamma_1, 1) = F(\gamma_0 * \beta_1) = F(\gamma_0) * F(\beta_1) = c(t) * (f(s), 1) = c * f_1$.

So $F \circ G$ is a path homotopy in $X \times I$ from $f_0 * c$ to $c * f_1$. Now consider $H \circ (F \circ G)$. Recall $H : X \times I \to Y$ and notice that $f_0 * c, c * f_1 \in X \times I$ (See Figure 58.3)

Lemma 58.4 Continued

Apply $H \circ (F \circ G)$ to $(\beta_0 * \gamma_1, 0)$ gives

$$\begin{aligned} H \circ (F \circ G)(\beta_0 * \gamma_1, 0) &= H(f_0 * c) \\ &= H(f_0) * H(c) = (H \circ f_0) * (H \circ c) \\ &= (h \circ f) * \alpha \text{ by } (\dagger) \text{ and } (\dagger\dagger\dagger) \end{aligned} \quad (11)$$

and

$$\begin{aligned} H \circ (F \circ G)(\beta_0 * \gamma_1, 1) &= H(c * f_1) \\ &= H(c) * H(f_1) = (H \circ c) * (H \circ f_1) \\ &= \alpha * (k \circ f) \text{ by } (\dagger\dagger) \text{ and } (\dagger\dagger\dagger) \end{aligned} \quad (12)$$

So $H \circ (F \circ G)$ is a path homotopy between $(h \circ f) * \alpha$ and $\alpha * (k \circ f)$. That is,

$$\begin{aligned} [\alpha * (k \circ f)] &= [(h \circ f) * \alpha] \text{ or} \\ [\alpha] * [k \circ f] &= [h \circ f] * [\alpha] \end{aligned} \quad (13)$$

This is the equation (*) that we needed to verify □

Corollary 58.5

Corollary 58.5. Let $h, k : X \to Y$ be homotopic continuous maps. Let $h(x_0) = y_0$ and $k(x_0) = y_1$. If induced homomorphism h_* is injective (one to one), surjective (onto), or trivial then so is the induced homomorphism k_*.
Proof. By Lemma 58.4, there exists a path from y_0 to y_1, such that $k_* = \hat{\alpha} \circ h_*$. By Theorem 52.1, $\hat{\alpha}$ is a group isomorphism and so is one to one and onto. So if h_* is one to one/onto/trivial, then so is k_*. □

Corollary 58.6

Corollary 58.6. Let $h : X \to Y$. If h is nulhomotopic, then h_* is the trivial homomorphism.
Proof. Let h be homotopic to a constant with $h(x_0) = y_0$. Define $i(x) = y_0$ for all $x \in X$, then by Theorem 58.4, we have that $\iota_* = \hat{\alpha} \circ h_*$ where α is a path from y_0 to y_0. The constant map ι induces the trivial homomorphism, so $\hat{\alpha} \circ h_*$ must be the trivial homomorphism and h_* must be trivial. □

Theorem 58.7

Theorem 58.7. Let $f : X \to Y$ be continuous. Let $f(x_0) = y_0$. If f is a homotopy equivalence, then the induced homomorphism

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0) \tag{14}$$

is an isomorphism.

Proof. Every homotopy is invertible, so let $g : Y \to X$ be a homotopy inverse for f. Consider the maps

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1) \tag{15}$$

where $x_1 = g(y_0)$ and $y_1 = f(x_1)$. The corresponding induced homomorphisms (which depend on the base point and this is indicated when necessary with subscript):

$$\pi_1(X, x_0) \xrightarrow{(f_{x_0})_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1) \xrightarrow{(f_{x_1})_*} \pi_1(Y, y_1) \tag{16}$$

Theorem 58.7

Since g is a homotopy inverse then

$$g \circ f : (X, x_0) \to (X, x_1) \tag{17}$$

is homotopic to the identity map, so by Lemma 58.4 there is a path α in X from x_0 to x_1 such that

$$(g \circ f)_* = \hat{\alpha} \circ (\iota_X)_* = \hat{\alpha} \tag{18}$$

since the identity map induces the identity group homomorphism. Now $\hat{\alpha}$ is a group isomorphism by Theorem 52.1, so $\hat{\alpha} = (g \circ f)_* = g_* \circ (f_{x_0})_*$ is an isomorphism.

Similarly, because $f \circ g$ is homotopic to the identity map ι_*, the homomorphism $(f \circ g)_* = (f_{x_1})_* \circ g_*$ is an isomorphism.

Theorem 58.7

The fact that $g_* \circ (f_{x_0})_*$ is an isomorophism implies that g_* is surjective (onto). The fact that $(f_{x_1})_* \circ g_*$ is an isomorphism implies that g_* is injective (one to one). So g_* is an isomorphism and hence so is $(g_*)^{-1}$. Since $g_* \circ (f_{x_0})_* = \hat{\alpha}$ from above, then $(f_{x_0})_* = (g_*)^{-1} \circ \hat{\alpha}$ is an isomorphism as claimed. □