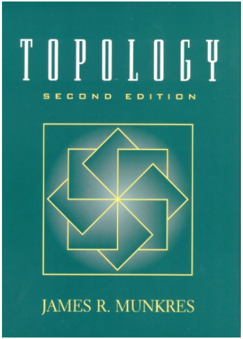
Introduction to Topology

Chapter 9. The Fundamental Group

Section 59. The Fundamental Group of S^n—Proofs of Theorems



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Theorem 59.1. Suppose $X = U \cup V$ where U and V are open sets of X. Suppose that $U \cap V$ is path connected and that $x_0 \in U \cap V$. Let i and j be the inclusion mappings of U and V, respectively, into X. Then the images of the induced homomorphisms

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \text{ and } j_* : \pi_1(V, x_0) \to \pi_1(X, x_0) \tag{1}$$

generate the group $\pi_1(X, x_0)$.

Proof. Recall that if group G is generated by elements $a_i \in G$ where $i \in I$, then the elements of G are all finite products of integer powers of the a_i (Fraleigh's Theorem 7.6). So the claim of this theorem is that any loop f in X based at x_0 is path homotopic to a product of the form $(g_1 * (g_2 * (\ldots * g_n)))$ where each g_i is a loop in X based at x_0 that lies either in U or V.

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Theorem 59.1

STEP 1 Choose a subdivision $0 = b_0 < b_1 < \ldots < b_m = 1$ of $[0, 1]$ such that for each i, the set $f([b_{i-1}, b_i])$ is contained in either U or V (which can be done since path f in X is compact) (Munkres cites the Lebesgue Number Lemma [ ],) If $f(b_i) \in U \cap V$ for all i, we stop. If not, let i be an index such that $f(b_i) \notin U \cap V$.

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Perform this process over each index value and the process yields a partition $0 = a_0 < a_1 < ... < a_n = 1$ of $[0, 1]$ such that $f(a_i) \in U \cap V$ for all i and $f([a_{i-1}, a_i])$ is contained either in U or in V for all i.

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STEP 2 Given f, let $0 = a_0 < a_1 < \ldots < a_n = 1$ be a partition of the sort constructed in STEP 1. Define f_i to be the path in X that equals the positive linear map of $[0,1]$ onto $[a_{i-1}, a_i]$ followed by f; so $f_i : [0,1] \to f|_{[a_{i-1}, a_i]}$.

So f_i is a path that lies either in U or in V, and by Theorem 51.3, $[f] = [f_1] * [f_2] * \ldots * [f_n]$.

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Now set $f_i = (\alpha_{i-1} * f_i) * \bar{\alpha}_i$ for each i. Then g_i is a loop in X based at x_0 whose image lies either in U or in V. Now

$$\begin{aligned}
[g_1] * [g_2] * [g_3] * \ldots * [g_n] =& [\alpha_0 * f_1 * \bar{\alpha_1}] * [\alpha_1 * f_2 * \bar{\alpha_2}] * [\alpha_2 * f_3 * \bar{\alpha_3}] * \\
& \ldots * [\alpha_{n-1} * f_n * \bar{\alpha_n}] \\
=& [\alpha_0] * [f_1] * [\bar{\alpha_1}] * [\alpha_1] * [f_2] * [\bar{\alpha_2}] * [\alpha_2] * [f_3] * \\
& [\bar{\alpha_3}] * \ldots * [\alpha_{n-1}] * [f_n] * [\bar{\alpha_n}] \textit{ by definition} \\
& \textit{of } [\alpha_{i-1} * f_i * \bar{\alpha_i}] \\
=& [f_1] * [f_2] * \ldots * [f_n] \\
=& [f]
\end{aligned} \tag{3}$$

So arbitrary path f is path homotopic to a product of loops g_i where each g_i is a loop in X based at x_0 whose image lies either in U or in V. That is, either $[g_i] \in \pi_1(U, x_0)$ or $[g_i] \in \pi_1(V, x_0)$ for all i. □

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Proof. By the definition of simply connected, we know that U and V are path connected and $\pi_1(U, x_0) \cong \pi_1(V, x_0) \cong \{e\}$ for some $x_0 \in U \cap V$. The hypothesis of Theorem 59.1 are satisfied and the images of i_* and j_* as given in Theorem 59.1 consist of the identity of $\pi_1(X, x_0) \cong \{e\}$. Since U and V are path connected and $U \cap V$ is nonempty, then $X = U \cup V$ is path connected. So by definition, X is simply connected. $\square$

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Proof. Let $\vec{p} = (0,0,0,1) \in \mathbb{R}^{n+1}$ and $\vec{q} = (0,0,...,-1)$ be the "north pole" and the "south pole" of S^n, respectively, where S^n is considered as embedded in $\mathbb{R}^{n+1}$ as

$$S^n = \{(x_1, x_2, .., x_{n+1}) | x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\}. \tag{4}$$

STEP 1 Define $f_i(S^n - \{\vec{p}\}) \to \mathbb{R}^n$ by the equation

$$f(\vec{x}) = f(x_1, ..., x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, ..., x_n). \tag{5}$$

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The map f is called the <u>stereographic projection</u>. (If we take the line in $\mathbb{R}^{n+1}$ through $\vec{p}$ and $\vec{x} \in S^n - \{\vec{p}\}$ then this line intersects the n-plane $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ at the point $f(\vec{x}) \times \{0\}$. This projection is used in complex analysis to map S^2 to the extended complex plane.)

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Consider the map $g : \mathbb{R}^n \to (S^n - \{\vec{p}\})$ given by

$$g(\vec{y}) = g(y_1, ..., y_n) = (t(y) \cdot y_1, ..., t(y) \cdot y_n, 1 - t(y)) \qquad (6)$$

where $t(y) = \frac{2}{(1+\|\vec{y}\|^2)}$. Then g is a left and right inverse of f. So f is a bijection, f is continuous on $S^n - \{\vec{p}\}$, and $f^{-1} = g$ is continuous on $\mathbb{R}^n$. So f is a homeomorphism between $S^n - \{\vec{p}\}$ and $\mathbb{R}^n$.

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Note that the reflection map $(x_1, .., x_n, x_{n+1}) \to (x_1, .., x_n, -x_{n+1})$ defines a homeomorphism of $S^n - \{\vec{p}\}$ with $S^n - \{\vec{q}\}$, so $S^n - \{\vec{q}\}$ is also homeomorphic to $\mathbb{R}^n$.

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<u>STEP 2</u> Let $U = S^n - \{\vec{p}\}$ and $V = S^n - \{\vec{q}\}$. Then U and V are open sets in S^n.

First, for $n \geq 1$ the sphere S^n is path connected since U and V are path connected (they are homeomorphic to $\mathbb{R}^n$ by STEP 1) and ahve the point $(1, 0, \ldots, 0)$ of S^n in common [for example].

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The space U and V are also simply connected, since they are homeomorphic to $\mathbb{R}^n$. $U \cap V = S^n \backslash \{\vec{p}, \vec{q}\}$, which is homeomorphic under stereographic projection to $\mathbb{R}^n \backslash \{(0,0)\}$ (since stereographic projection maps $\vec{q}$ to $(0,0)$).

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