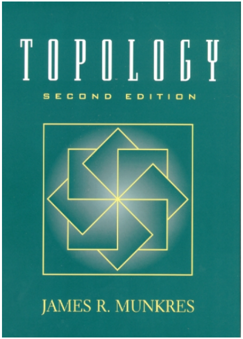
Introduction to Topology

Chapter 9. The Fundamental Group

Section 60. Fundamental Groups of Some Surfaces—Proofs of Theorems



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Theorem 60.1

Theorem 60.1. Let X and Y be topological spaces and $x_0 \in X$, $y_0 \in Y$. The group $\pi_1(X \times Y, x_0 \times y_0)$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projection mappings. Since p and q are continuous, we have the induced homomrophisms

$$\begin{aligned} p_* :& \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \\ q_* :& \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(Y, y_0) \end{aligned} \tag{1}$$

By the statement above, the map $\Phi : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ defined by the equation

$$\Phi([f]) = p_*([f]) \times q_*([f]) = [p \circ f] \times [q \circ f] \textit{ by the defn of } p_*,\ q_* \tag{2}$$

is a homomorphism. We now show that Φ is in fact an isomorphism.

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Let $g : I \longrightarrow X$ be a loop in X based at x_0. Let $h : I \longrightarrow Y$ be a loop in Y based at y_0. $[g] \times [h] \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$. Define $f : I \longrightarrow X \times Y$ by the equation $f(s) = g(s) \times h(s)$. Then f is a loop in $X \times Y$ (f is continuous) based at $x_0 \times y_0$, and $\Phi([f]) = [p \circ f] \times [q \circ f] = [g] * [h]$ and so Φ is onto.

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Suppose that $f : I \to X \times Y$ is a loop in $X \times Y$ based at (x_0, y_0) and that

$$\Phi([f]) = [p \circ f] \times [q \circ f] \tag{3}$$

is the identity element of $\pi_1(X, x_0) \times \pi_1(Y, y_0)$. The identity of $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ is $[(x_0, y_0)]$, so $p \circ f \cong_p e_{x_0}$ and $q \circ f \cong_p e_{y_0}$.

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So let G be a path homotopy from $p \circ f$ to e_{x_0} and let H be a path homotopy from $q \circ f$ to e_{y_0}. Define $F(s,t) = G(s,t) \times H(s,t)$. Then $F : I \times I \to X \times Y$ is a path homotopy from f to the constant loop at (x_0, y_0). So if $\Phi([f])$ is the identity in $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ then $[f]$ is the identity in $\pi_1(X \times Y, (x_0, y_0))$.

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Theorem 60.3

Theorem 60.3. The projective plane P^2 is a surface, and the quotient map $p : S^2 \to P^2$ defined as $p(\vec{x}) = [\vec{x}] = \{-\vec{x}, \vec{x}\}$ is a covering map.

Proof. First we show that p is an open map. (i.e., p maps opens sets to open sets). Let U be open in S^2. The antipodal map $a : S^2 \to S^2$ given by $a(\vec{x}) = -\vec{x}$ is clearly a homeomorphism of S^2. Hence $a(U)$ is open in S^2.

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So $p^{-1}(p(U))$ is the set of all elements of S^2 that are contained in an equivalence class of $p(U)$. So $p^{-1}(p(U)) = U \cup a(U)$. So $p^{-1}(p(U))$ is open.

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Theorem 60.3 Continued

With S^2_N and S^2_S as described above, let

$$U_N = U \cap S^2_N \ \ and \ \ U_S = U \cap S^2_S \tag{5}$$

Then U_N is open relative to S^2_N and U_S is open relative to S^2_S. With P^2 as represented as S^2_N as described above, we have that $p|_{S^2_N}$ is the embedding mapping of S^2_N to P^2, and $p|_{S^2_S}$ is the mapping $a : S^2_S \to S^2_N$ followed by the embedding mapping.

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Now we show that p is a covering map. For any $\vec{y} \in P^2$, choose $\vec{x} \in p^{-1}(\vec{y})$ and choose an ϵ-neighborhood U of $\vec{x}$ in S^2 for some $0 < \epsilon < 1$ using the Euclidean metric of $\mathbb{R}^3$. Then U contains no pair $\{\vec{z}, a(\vec{z})\}$ of antipodal points of S^2 since $d(\vec{z}, a(\vec{z})) = 2$. As a result, the map $p : U \to p(U)$ is one to one and onto (i.e., onto $p(U)$).

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Now we show that p is a covering map. For any $\vec{y} \in P^2$, choose $\vec{x} \in p^{-1}(\vec{y})$ and choose an ϵ-neighborhood U of $\vec{x}$ in S^2 for some $0 < \epsilon < 1$ using the Euclidean metric of $\mathbb{R}^3$. Then U contains no pair $\{\vec{z}, a(\vec{z})\}$ of antipodal points of S^2 since $d(\vec{z}, a(\vec{z})) = 2$. As a result, the map $p : U \to p(U)$ is one to one and onto (i.e., onto $p(U)$). Now p is continuous and is open by above, so it is a homeomorphism between U and $p(U)$ (since p is bijective, then p^{-1} exists and since p maps open sets to open sets, then p^{-1} is continuous).

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Theorem 60.3 Continued

Since S^2 is a surface, then it has a countable basis $\{U_n\}$, then $\{p(U_n)\}$ is a countable basis of P^2, since p is a homeomorphism.

Now to show that P^2 is Hausdorff. Let $y_1, y_2 \in P^2$ ($y_1 \neq y_2$). The set $p^{-1}(y_1) \cup p^{-1}(y_2)$ consists of four points in S^2. Let 2ϵ be the minimum distance between pairs of them.

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Theorem 60.3 Continued

Since p is a homeomorphism from S^2 to P^2, every point $\vec{x}$ of P^2 has a neighborhood homeomorphic with an open subset of S^2. Since S^2 is a surface then every point in S^2 has a neighborhood homeomorphic to an open subset of $\mathbb{R}^2$, and so every open subset of S^2 is homeomorphic to an open subset of $\mathbb{R}^2$. By taking this open subset fo $\mathbb{R}^2$ into S^2 and then P^2, we get a neighborhood of $\vec{x}$ [illegible] P^2 that is homeomorphic with an open set in $\mathbb{R}^2$. So P^2 is a surface. □

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Corollary 60.4

Corollary 60.4. $\pi_1(P^2, y)$ is a group of order 2.

Proof. The projection $p : S^2 \to P^2$ is a covering map by Theorem 60.3. Since S^2 is simply connected (Theorem 59.3), by Theorem 54.4 there is a bijective correspondence between $\pi_1(P^2, y)$ and the set $p^{-1}(y)$. Since $p^{-1}(y)$ is a two element set, then $\pi_1(P^2, y)$ is a group of order 2. □

Lemma 60.5

Lemma 60.5. The fundamental group of the figure eight is not abelian.

Proof. Let X be the union of two circls A and B in $\mathbb{R}^2$ with intersection x_0. We give a covering space E of X.

For space E, take the subspace of $\mathbb{R}^2$ consisting of the x-axis, the y-axis, and circles tangent to these axes, one circle tangent to the x-axis at each nonzero integer point and lying above the x-axis and one circle tangent to the y-axis at each nonzero integer point and lying to the right of the y-axis.

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Lemma 60.5 Continued

We now describe the relevant mappings geometrically instead of quantitatively. The projection map $p : E \to X$ wraps the x-axis around the circle A and wraps the y-axis around the other circle B (with $(0, 0)$ mapped to the point x_0 common to both A and B).

Similar to the mapping for S^1, map each integer point to the base point x_0. Each circle tangent to an integer point on the x-axis is mapped homeomorphically by p onto B. Each circle tangent to an integer point on the y-axis is mapped homeomorphically onto A.

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In each case the point of tangency is mapped onto the point x_0. This mapping $p : E \to X$ is indeed a covering map (Munkres leaves "it to you to check mentally" this claim).

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In each case the point of tangency is mapped onto the point x_0. This mapping $p : E \to X$ is indeed a covering map (Munkres leaves "it to you to check mentally" this claim).

Lemma 60.5 Continued

Now let $\tilde{f} : I \to E$ be the path $\tilde{f}(s) = \{s\} \times \{0\}$, going along the x-axis from the origin to point $(1, 0)$. Let $\tilde{g} : I \to E$ be the path $\tilde{g}(s) = \{0\} \times \{s\}$ going along the y-axis from the origin to the point $(0, 1)$. Let $f = p \circ \tilde{f}$ and $g = p \circ \tilde{g}$.

Then f and g are loops in the figure eight based at x_0. We have informally described p, so WLOG say f goes around circle A once counterclockwise and g goes around the circle B once counterclockwise.

Lemma 60.5 Continued

Now let $\tilde{f} : I \to E$ be the path $\tilde{f}(s) = \{s\} \times \{0\}$, going along the x-axis from the origin to point $(1, 0)$. Let $\tilde{g} : I \to E$ be the path $\tilde{g}(s) = \{0\} \times \{s\}$ going along the y-axis from the origin to the point $(0, 1)$. Let $f = p \circ \tilde{f}$ and $g = p \circ \tilde{g}$.

Then f and g are loops in the figure eight based at x_0. We have informally described p, so WLOG say f goes around circle A once counterclockwise and g goes around the circle B once counterclockwise.

Lemma 60.5 Continued

We now show that $f * g$ and $g * f$ are noth path homotopic in the figure eight. Lift the path $f * g$ to E to a path in E starting at $e_0 = (0,0)$. Then $f * g$ lifts to a path in E from $(0,0)$ to $(1,0)$ and then goes around the circle tangent to the x-axis at $(1,0)$ and ends at $(1,0) \in E$.

The path $g * f$ lifts to a path in E from $(0,0)$ to $(0,1)$ and then goes around the circle tangent to the y-axis at $(0,1)$ and ends at $(0,1)$. By Theorem 54.3, since the lifts of $f * g$ and $g * f$ (denoted $f \tilde{*} g$ and $g \tilde{*} f$) do not end at the same point of E, then $f * g$ is not path homotopic to $g * f$.

Lemma 60.5 Continued

We now show that $f * g$ and $g * f$ are noth path homotopic in the figure eight. Lift the path $f * g$ to E to a path in E starting at $e_0 = (0, 0)$. Then $f * g$ lifts to a path in E from $(0, 0)$ to $(1, 0)$ and then goes around the circle tangent to the x-axis at $(1, 0)$ and ends at $(1, 0) \in E$.

The path $g * f$ lifts to a path in E from $(0, 0)$ to $(0, 1)$ and then goes around the circle tangent to the y-axis at $(0, 1)$ and ends at $(0, 1)$. By Theorem 54.3, since the lifts of $f * g$ and $g * f$ (denoted $f \tilde{*} g$ and $g \tilde{*} f$) do not end at the same point of E, then $f * g$ is not path homotopic to $g * f$.

That is, $[f * g] \neq [g * f]$, or (by definition of $*$ in the fundamental group) $[f] * [g] \neq [g] * [f]$. Since $[f]$, $[g] \in \pi_1$(figure eight, b_0), then the fundamental group of the figure eight is not abelian. $\square$

Lemma 60.5 Continued

We now show that $f * g$ and $g * f$ are noth path homotopic in the figure eight. Lift the path $f * g$ to E to a path in E starting at $e_0 = (0, 0)$. Then $f * g$ lifts to a path in E from $(0, 0)$ to $(1, 0)$ and then goes around the circle tangent to the x-axis at $(1, 0)$ and ends at $(1, 0) \in E$.

The path $g * f$ lifts to a path in E from $(0, 0)$ to $(0, 1)$ and then goes around the circle tangent to the y-axis at $(0, 1)$ and ends at $(0, 1)$. By Theorem 54.3, since the lifts of $f * g$ and $g * f$ (denoted $f \tilde{*} g$ and $g \tilde{*} f$) do not end at the same point of E, then $f * g$ is not path homotopic to $g * f$.

That is, $[f * g] \neq [g * f]$, or (by definition of $*$ in the fundamental group) $[f] * [g] \neq [g] * [f]$. Since $[f], [g] \in \pi_1(\text{figure eight}, b_0)$, then the fundamental group of the figure eight is not abelian. □

Theorem 60.6

Theorem 60.6. The fundamental group of the double torus is not abelian.
Proof. The double torus, denoted $T \# T$ is, informally

$$j_*([f]) * j_*([g]) \neq j_*([g]) * j_*([f]) \tag{6}$$

Where $j_*([f]), j_*([g]) \in \pi_1(T \# T, x_0)$.

That is, the fundamental group of $T \# T$ is nonabelian. □