Chapter 2. Topological Spaces and Continuous Functions

Section 12. Topological Spaces

Note. Recall from your senior level analysis class that a set U of real numbers is defined to be *open* if for any $u \in U$ there is $\varepsilon > 0$ such that $(u-\varepsilon, u+\varepsilon) \subset U$. The open sets of real numbers satisfy the following three properties:

(1) $\varnothing$ and $\mathbb{R}$ are open.

(2) The union of any collection of open sets is open.

(3) The intersection of a finite collection of open sets is open.

For more details, see my notes from Analysis 1 (MATH 4217/5217) on "Topology of the Real Numbers":

`http://faculty.etsu.edu/gardnerr/4217/notes/3-1.pdf`

Note. The definition of an open set of real numbers is based on our ability to measure distance in $\mathbb{R}$ (namely, it uses the *metric* on $\mathbb{R}$). Analogously, in any metric space the metric can similarly be used to define an open set. Our purpose is to define an open set abstractly by mimicking the three properties which open sets of real numbers (and open sets in any metric space) satisfy. Hence we have the following definition.

Definition. A *topology* on a point set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$.

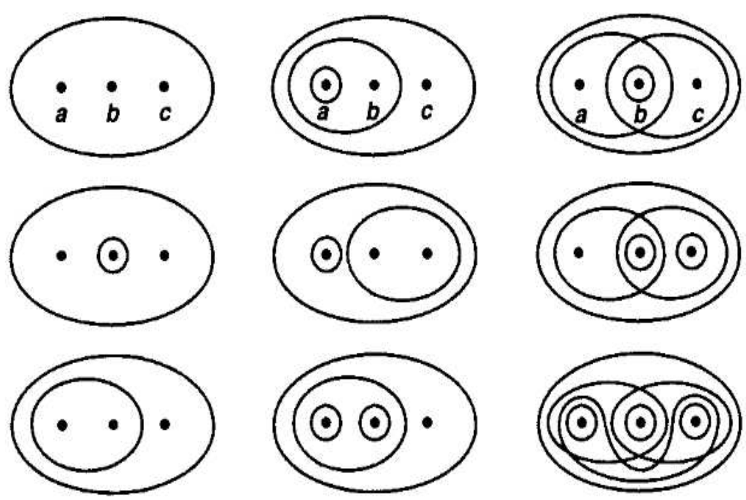
(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$; that is, if $\{U_\alpha\}_{\alpha \in A} \subset \mathcal{T}$ then $\cup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$ is in $\mathcal{T}$; that is, if $U_1, U_2, \ldots, U_n \in \mathcal{T}$ then $\cap_{i=1}^n U_i \in \mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is a *topological space.* A subset of X which is in $\mathcal{T}$ is called an *open set.*

Example. With $X = \mathbb{R}$ and $\mathcal{T} = \{U \subset \mathbb{R} \mid U$ is an open set of real numbers as defined above$\}$ is a topological space. It is $\mathbb{R}$ under the *standard topology.*

Example 1. Let $X = \{a, b, c\}$. Then there are 9 possible topologies on X. They are illustrated as follows where sets are represented by enclosing their elements with a curve:



Example 2. If X is any set and $\mathcal{T}_1$ is the collection of all subsets of X (that is, $\mathcal{T}_1$ is the power set of X, $\mathcal{T}_1 = \mathcal{P}(X)$) then this is a topological spaces. $\mathcal{T}$ is called the *discrete topology* on X. At the other extreme is the topology $\mathcal{T}_2 = \{\varnothing, X\}$, called the *trivial topology* on X.

Example 3. Let X be a set and $\mathcal{T}_f$ be the collection of all subsets U of X such that $X \setminus U = \{x \in X \mid x \notin U\}$ is either finite or all of X. Then $\mathcal{T}_f$ is a topology on X, called the *finite complement topology*. See page 77 for the proof that this is in fact a topology.

Example 4. Let X be a set and $\mathcal{T}_c$ be the collection of all subsets U of X such that $X \setminus U$ is either countable or all of X. Then $\mathcal{T}_c$ is a topology on X, as you can verify for homework.

Definition. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supset \mathcal{T}$ then $\mathcal{T}'$ is *finer* then $\mathcal{T}$; if $\mathcal{T}'$ properly contains $\mathcal{T}$ then $\mathcal{T}'$ is *strictly finer* than $\mathcal{T}$. We also define in these cases that $\mathcal{T}$ is *coarser* then $\mathcal{T}'$, or $\mathcal{T}$ is *strictly coarser* then $\mathcal{T}'$, respectively. We define "$\mathcal{T}$ is *comparable* to $\mathcal{T}'$" if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$.

Note. If $\mathcal{T}'$ is finer than $\mathcal{T}$ then $\mathcal{T}'$ has more open sets than $\mathcal{T}$. We will see that we define continuity and limits of functions and sequences in terms of the behavior of all open sets satisfying certain properties. Therefore, the more open sets we have in a topological space, the "harder" it is for a function to be continuous or a limit to exist. For this reason, analysts often replace the term "finer" with "stronger" and replace "coarser" with "weaker" (see my notes for Real Analysis 2 at `http://faculty.etsu.edu/gardnerr/5210/notes/11-4.pdf`). However, topologists tend to interchange these uses of the words "stronger" and "weaker" (see Munkres comment on page 78).

Revised: 5/28/2016