Section 13. Basis for a Topology

Note. In this section, we consider a basis for a topology on a set which is, in a sense, analogous to the basis for a vector space. Whereas a basis for a vector space is a set of vectors which (efficiently; i.e., linearly independently) generates the whole space through the process of raking linear combinations, a basis for a topology is a collection of open sets which generates all open sets (i.e., elements of the topology) through the process of taking unions (see Lemma 13.1).

Definition. Let $X$ be a set. A basis for a topology on $X$ is a collection $\mathcal{B}$ of subsets of $X$ (called basis elements) such that

1. For each $x \in X$, there is at least one basis element $B \in \mathcal{B}$ such that $x \in B$.

2. If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$ then there is $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_2 \cap B_2$.

The topology $\mathcal{T}$ generated by $\mathcal{B}$ is defined as: A subset $U \subset X$ is in $\mathcal{T}$ if for each $x \in U$ there is $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. (Therefore each basis element is in $\mathcal{T}$.)

Note. We need to prove that the alleged topology generated by basis $\mathcal{B}$ is really in fact a topology.
Theorem 13.A. Let $\mathcal{B}$ be a basis for a topology on $X$. Define

$$\mathcal{T} = \{U \subseteq X \mid x \in U \text{ implies } x \in B \subseteq U \text{ for some } B \in \mathcal{B}\};$$

the “topology” generated be $\mathcal{B}$. Then $\mathcal{T}$ is in fact a topology on $X$.

Example. A set of real numbers (under the standard topology) is open if and only if it is a countable disjoint union of open intervals. This is one of the most important results from Analysis 1 (MATH 4217/5217)! A largely self-contained proof of this (only requiring a knowledge of lub and glb of a set of real numbers) can be found in my supplemental notes to Analysis 1 at:


So a basis for the standard topology on $\mathbb{R}$ is given by the set of all open intervals of real numbers:

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}, a < b\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\} \cup \{(a, \infty) \mid a \in \mathbb{R}\}.$$

In fact, a countable basis for the standard topology is given by $\mathcal{B}' = \{(a, b) \mid a, b \in \mathbb{Q}, a < b\}$. This is based in part on the fact that a countable union of countable sets is countable (see Munkres’ Theorem 7.5). See Exercise 13.8(a).

Example 1. A basis for the standard topology on $\mathbb{R}^2$ is given by the set of all circular regions in $\mathbb{R}^2$:

$$\mathcal{B} = \{B((x_0, y_0), r) \mid r > 0 \text{ and } B((x_0, y_0), r) = \{(x, y) \in \mathbb{R}^2 \mid (x-x_0)^2 + (y-y_0)^2 < r^2\}\}.$$

In fact, a countable basis is similarly given by considering all $B((p_0, q_0), r)$ where $p_0, q_0 \in \mathbb{Q}$ and $r \in \mathbb{Q}$ where $r > 0$. 
Example 2. A basis for the standard topology on $\mathbb{R}^2$ is also given by the set of all open rectangular regions in $\mathbb{R}^2$ (see Figure 13.2 on page 78).

Example 3. If $X$ is any set, $\mathcal{B} = \{ \{x\} \mid x \in X \}$ is a basis for the discrete topology on $X$.

Note. The following result makes it more clear as to how a basis can be used to build all open sets in a topology.

Lemma 13.1. Let $X$ be a set and let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on $X$. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Note. The previous result allows us to create (“generate”) a topology from a basis. The following result allows us to test a collection of open sets to see if it is a basis for a given topology.

Lemma 13.2. Let $(X, \mathcal{T})$ be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of $X$ such that for each open subset $U \subseteq X$ and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$ on $X$.

Note. The following lemma allows us to potentially compare the fineness/coarseness to two topologies on set $X$ based on properties of respective bases.
**Lemma 13.3.** Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on $X$. Then the following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.

2. For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing $x$, there is a basis element $B' \in \mathcal{B}$ such that $x \in B' \subset B$.

**Note.** We now define three topologies on $\mathbb{R}$, one of which (the “standard topology”) should already be familiar to you.

**Definition.** Let $\mathcal{B}$ be the set of all open bounded intervals in the real line:

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}, a < b\}.$$  

The topology generated by $\mathcal{B}$ is the *standard topology* on $\mathbb{R}$.

**Definition.** Let $\mathcal{B}'$ be the set of all half open bounded intervals as follows:

$$\mathcal{B}' = \{[a, b) \mid a, b \in \mathbb{R}, a < b\}.$$  

The topology generated by $\mathcal{B}'$ is the *lower limit topology* on $\mathbb{R}$, denoted $\mathbb{R}_\ell$.

**Definition.** Let $K = \{1/n \mid n \in \mathbb{N}\}$. Let

$$\mathcal{B}'' = \{(a, b) \mid a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K \mid a, b \in \mathbb{R}, a < b\}.$$  

The topology generated by $\mathcal{B}''$ is the *K-topology* on $\mathbb{R}$, denoted $\mathbb{R}_K$. 
Note. The relationship between these three topologies on \( \mathbb{R} \) is as given in the following.

**Lemma 13.4.** The topologies of \( \mathbb{R}_\ell \) and \( \mathbb{R}_K \) are each strictly finer than the standard topology on \( \mathbb{R} \), but are not comparable with one another.

**Definition.** A subbasis \( S \) for a topology on set \( X \) is a collection of subsets of \( X \) whose union equals \( X \). The topology generated by the subbasis \( S \) is defined to be the collection \( T \) of all unions of finite intersections of elements of \( S \).

Note. Of course we need to confirm that the topology generated by a subbasis is in fact a topology.

**Theorem 13.B.** Let \( S \) be a subbasis for a topology on \( X \). Define \( T \) to be all unions of finite intersections of elements of \( S \). Then \( T \) is a topology on \( X \).

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