Section 13. Basis for a Topology

Note. In this section, we consider a basis for a topology on a set which is, in a sense, analogous to the basis for a vector space. Whereas a basis for a vector space is a set of vectors which (efficiently; i.e., linearly independently) generates the whole space through the process of raking linear combinations, a basis for a topology is a collection of open sets which generates all open sets (i.e., elements of the topology) through the process of taking unions (see Lemma 13.1).

Definition. Let X be a set. A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X (called *basis elements*) such that

(1) For each $x \in X$, there is at least one basis element $B \in \mathcal{B}$ such that $x \in B$.

(2) If $x \in B_a \cap B_2$ where $B_1, B_2 \in \mathcal{B}$ then there is $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_2 \cap B_2$.

The topology $\mathcal{T}$ *generated by* $\mathcal{B}$ is defined as: A subset $U \subset X$ is in $\mathcal{T}$ if for each $x \in U$ there is $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. (Therefore each basis element is in $\mathcal{T}$.)

Note. We need to prove that the alleged topology generated by basis $\mathcal{B}$ is really in fact a topology.

Theorem 13.A. Let $\mathcal{B}$ be a basis for a topology on X. Define

$$\mathcal{T} = \{U \subset X \mid x \in U \text{ implies } x \in B \subset U \text{ for some } B \in \mathcal{B}\},$$

the "topology" generated be $\mathcal{B}$. Then $\mathcal{T}$ is in fact a topology on X.

Example. A set of real numbers (under the standard topology) is open if and only if it is a countable disjoint union of open intervals. This is one of the most important results from Analysis 1 (MATH 4217/5217)! A largely self-contained proof of this (only requiring a knowledge of lub and glb of a set of real numbers) can be found in my supplemental notes to Analysis 1 at:

`http://faculty.etsu.edu/gardnerr/4217/notes/Supplement-Open-Sets.pdf`

So a basis for the standard topology on $\mathbb{R}$ is given by the set of all open intervals of real numbers:

$$\mathcal{B} = \{(a,b) \mid a,b \in \mathbb{R}, a < b\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\} \cup \{(a,\infty) \mid a \in \mathbb{R}\}.$$

In fact, a countable basis for the standard topology is given by $\mathcal{B}' = \{(a,b) \mid a,b \in \mathbb{Q}, a < b\}$. This is based in part on the fact that a countable union of countable sets is countable (see Munkres' Theorem 7.5). See Exercise 13.8(a).

Example 1. A basis for the standard topology on $\mathbb{R}^2$ is given by the set of all circular regions in $\mathbb{R}^2$:

$$\mathcal{B} = \{B((x_0,y_0),r) \mid r > 0 \text{ and } B((x_0,y_0),r) = \{(x,y) \in \mathbb{R}^2 \mid (x-x_0)^2+(y-y_0)^2 < r^2\}\}.$$

In fact, a countable basis is similarly given by considering all $B((p_0,q_0),r)$ where $p_0, q_0 \in \mathbb{Q}$ and $r \in \mathbb{Q}$ where $r > 0$.

Example 2. A basis for the standard topology on $\mathbb{R}^2$ is also given by the set of all open rectangular regions in $\mathbb{R}^2$ (see Figure 13.2 on page 78).

Example 3. If X is any set, $\mathcal{B} = \{\{x\} \mid x \in X\}$ is a basis for the discrete topology on X.

Note. The following result makes it more clear as to how a basis can be used to build all open sets in a topology.

Lemma 13.1. Let X be a set and let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Note. The previous result allows us to create ("generate") a topology from a basis. The following result allows us to test a collection of open sets to see if it is a basis for a given topology.

Lemma 13.2. Let $(X, \mathcal{T})$ be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open subset $U \subset X$ and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$ on X.

Note. The following lemma allows us to potentially compare the fineness/coarseness to two topologies on set X based on properties of respective bases.

Lemma 13.3. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$.

(2) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}$ such that $x \in B' \subset B$.

Note. We now define three topologies on $\mathbb{R}$, one of which (the "standard topology") should already be familiar to you.

Definition. Let $\mathcal{B}$ be the set of all open bounded intervals in the real line:

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}, a < b\}.$$

The topology generated by $\mathcal{B}$ is the *standard topology* on $\mathbb{R}$.

Definition. Let $\mathcal{B}'$ be the set of all half open bounded intervals as follows:

$$\mathcal{B}' = \{[a, b) \mid a, b \in \mathbb{R}, a < b\}.$$

The topology generated by $\mathcal{B}'$ is the *lower limit topology* on $\mathbb{R}$, denoted $\mathbb{R}_\ell$.

Definition. Let $K = \{1/n \mid n \in \mathbb{N}\}$. Let

$$\mathcal{B}'' = \{(a, b) \mid a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K \mid a, b \in \mathbb{R}, a < b\}.$$

The topology generated by $\mathcal{B}''$ is the *K-topology* on $\mathbb{R}$, denoted $\mathbb{R}_K$.

Note. The relationship between these three topologies on $\mathbb{R}$ is as given in the following.

Lemma 13.4. The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are each strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

Definition. A *subbasis* $\mathcal{S}$ for a topology on set X is a collection of subsets of X whose union equals X. The *topology generated by the subbasis* $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

Note. Of course we need to confirm that the topology generated by a subbasis is in fact a topology.

Theorem 13.B. Let $\mathcal{S}$ be a subbasis for a topology on X. Define $\mathcal{T}$ to be all unions of finite intersections of elements of $\mathcal{S}$. Then $\mathcal{T}$ is a topology on X.

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