Section 14. The Order Topology

Note. Munkres defines an order relation (which he refers to in this section as a "simple order"), denoted "<," on a set A as a relation (see page 21) satisfying:

(1) Comparability: For every $x, y \in A$ for which $x \neq y$, either $x < y$ or $y < x$.

(2) Nonreflexivity: For no $x \in A$ does the relation $x < x$ hold.

(3) Transitivity: If $x < y$ and $y < z$ then $x < z$.

In this section, we use a simple order relation on a set to define a topology on the set.

Definition. Let X be a set. A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X (called *basis elements*) such that

(1) For each $x \in X$, there is at least one basis element $B \in \mathcal{B}$ such that $x \in B$.

(2) If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$ then there is $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_2 \cap B_2$.

The topology $\mathcal{T}$ *generated by* $\mathcal{B}$ is defined as: A subset $U \subset X$ is in $\mathcal{T}$ if for each $x \in U$ there is $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. (Therefore each basis element is in $\mathcal{T}$.)

Definition. Let X be a set with a simple order relation $<$. The following sets are *intervals* in X:

$$(a, b) = \{x \in X \mid a < x < b\} \text{ (open intervals)}$$
$$(a, b] = \{x \in X \mid a < x \leq b\} \text{ (half-open intervals)}$$
$$[a, b) = \{x \in X \mid a \leq x < b\} \text{ (half-open intervals)}$$
$$[a, b] = \{x \in X \mid a \leq x \leq b\} \text{ (closed intervals).}$$

Definition. Let X be a set with a simple order relation and assume X hax more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_0, b)$ where a_0 is the least element (if one exists) of X.

(3) All intervals of the form $(a, b_0]$ where b_0 is the greatest element (if one exists) of X.

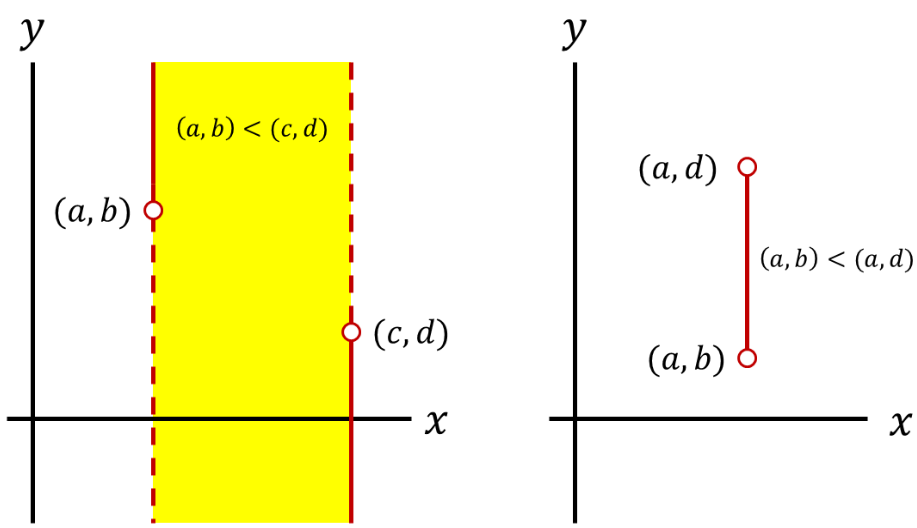
The collection $\mathcal{B}$ is a basis for a topology on X called the *order topology*.

Note. Of course we must verify that $\mathcal{B}$ really is a basis for a topology.

Theorem 14.A. Let X be a set with a simple order relation and let $\mathcal{B}$ consist of all open intervals (a, b), all intervals $[a_0, b)$, and all intervals $(a, b_0]$, where a_0 is the least element of X and b_0 is the greatest element of X (if such exist). Then $\mathcal{B}$ is a basis for a topology on X.

Example 1. The standard topology on $\mathbb{R}$ is the order topology based on the usual "less than" order on $\mathbb{R}$.

Example 2. We can put a simple order relation on $\mathbb{R}^2$ as follows: $(a,b) < (c,d)$ if either (1) $a < c$, or (2) $a = c$ and $b < d$. This is often called the lexicographic ordering (see my Complex Analysis 1 [MATH 5510] notes for a mention on the lexicographic ordering applied to $\mathbb{C}$: `http://faculty.etsu.edu/gardnerr/5510/Ordering-C.pdf`) or, as Munkres calls it, the dictionary order. These two types of open intervals under this simple order relation are then as follows:



Notice that this can easily be generalized to $\mathbb{R}^n$.

Example 4. Let $X = \{1,2\} \times \mathbb{N}$ with the dictionary order. Then $(1,1)$ is the least element of X, though there is no greatest element of X. The ordering produces the inequalities: $(1,1) < (1,2) < (1,3) < \cdots < (2,1) < (2,2) < \cdots$ where the first "$\cdots$" indicates that all elements of the form $(1,n)$ are present. Notice that all but one singleton is in the basis $\mathcal{B}$ for the order topology:

$$(1,1) = [(1,1),(1,2),$$
$$(1,n) = (1,n-1),(1,n+1)) \text{ for } n > 1,$$
$$(2,n) = (2,n-1),(2,n+1)) \text{ for } n > 1,$$

but a basis element containing $(2,1)$ must be of the form (a,b) where $a < (2,1)$ and $(2,1) < b$. But then a is of the form $(1,n)$ for some $n \in \mathbb{N}$, so (a,b) contains an infinite number of elements of X less than $(2,1)$. Now any open set containing $(2,1)$ must contain a basis element about $(2,1)$ and so the singleton $(2,1)$ is the lone singleton in the topological space which is not open.

Definition. If X is a set with a simple order relation $<$, and $a \in X$ then there are four subsets of X, called *rays* determined by a. They are the following:

$$(a,+\infty) = \{x \in X \mid x > a\}$$
$$(-\infty,a) = \{x \in X \mid x < a\}$$
$$[a,+\infty) = \{x \in X \mid x \geq a\}$$
$$(-\infty,a) = \{x \in X \mid x \leq a\}.$$

The first two types of rays are called *open rays* and the last two types are called *closed rays*.

Note. The open rays in X are open sets in the order topology since:

(1) If X has a greatest element b_0 then $(a, +\infty) = (a, b_0] \in \mathcal{B}$ is given.

(2) If X does not have a greatest element then $(a, +\infty) = \sup_{x>a}(a, x)$ is open.

(3) If X has a least element a_0 then $(-\infty, a) = [a_0, a) \in \mathcal{B}$ is open.

(4) If X does not have a least element then $(-\infty, a) = \cup_{x<a}(x, a)$ is open.

Notice that we have not yet defined "closed set," but we will in Section 17.

Theorem 14.B. Let X be a set with a simple order relation. The open rays form a subbasis for the order topology $\mathcal{T}$ on X.

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