# **Section 15.** The Product Topology on $X \times Y$

**Note.** If $X$ and $Y$ are topological spaces, then there in a natural topology on the Cartesian product set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$. In Section 19, we study a more general product topology.

**Definition.** Let $X$ and $Y$ be topological spaces. The *product topology* on set $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where $U$ is an open subset of $X$ and $V$ is an open subset of $Y$.

**Note.** As usual, we need to confirm that Munkres' definition is meaningful and so we must verify that $\mathcal{B}$ is a basis or a topology. Since $X$ is pen and $Y$ is open, then $X \times Y \in \mathcal{B}$ is open and part (1) of the definition of "basis" is satisfied. For part (2) of the definition, let $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$ be elements of $\mathcal{B}$, then $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$ is a basis element because $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in $X$ and $Y$, respectively (see Figure 15.1 on page 87).

**Note.** The following result tells us how to find a basis for the product topology on $X \times Y$ in terms of bases of $X$ and $Y$.

**Theorem 15.1.** If $\mathcal{B}$ is a basis for the topology of $X$ and $\mathcal{C}$ is a basis for the topology of $Y$, then the collection $\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$.

**Example 1.** The standard topology on $\mathbb{R}^2$ is the product topology on $\mathbb{R} \times \mathbb{R}$ where we have the standard topology on $\mathbb{R}$. Since a basis for the standard topology on $\mathbb{R}$ is $\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}, a < b\}$ (by the definition of "standard topology on $\mathbb{R}$"), then Theorem 15.1 implies that a basis for the standard topology on $\mathbb{R} \times \mathbb{R}$ is

$$\mathcal{B}'\{(a, b) \times (c, d) \mid a, b, c, d \in \mathbb{R}, a < b, c < d\}.$$

**Note.** The following definition introduces projections which allow us to move sets from $X \times Y$ into the constituent sets $X$ and $Y$.

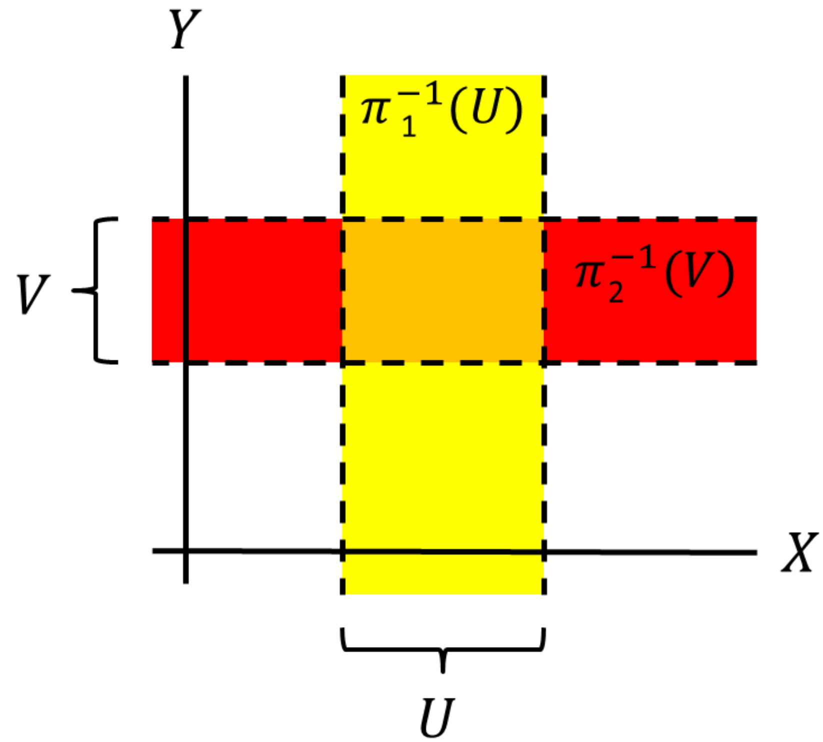
**Definition.** Let $\pi_1 : X \times Y \to X$ be defined (pointwise) by the equation $\pi_1((x, y)) = x$. Let $\pi_2 : X \times Y \to Y$ be defined by the equation $\pi_2((x, y)) = y$. The maps $\pi_1$ and $\pi_2$ are the *projections* of $X \times Y$ onto its first and second factors, respectively.

**Note.** For any $U \subset X$, we can consider the inverse image of $U$ under $\pi_1$:

$$\pi_1^{-1}(U) = \{(x, y) \in X \times Y \mid \pi_1((x, y)) \in U\} = \{(x, y) \in X \times Y \mid x \in U\} = U \times Y.$$

(Notice that this involves inverse *images* and not inverse *functions*, so there is no concern about one to one-ness.) Similarly, for $V \subset Y$ we have $\pi_2^{-1}(V) = X \times V$. For

$U$ and $V$ open bounded intervals in $\mathbb{R}$, we get the following visual representation of $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$ in $\mathbb{R} \times \mathbb{R}$:




**Note.** The inverse images of open sets in $X$ and $Y$ under projections can be used to create a subbasis for the product topology on $X \times Y$ as follows.

**Theorem 15.2.** The set

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.