Section 16. The Subspace Topology

Note. Recall from Analysis 1 that a set of real numbers U is *open relative to set* X if there is an open set of real numbers $\mathcal{O}$ such that $U = X \cap \mathcal{O}$ (see page 5 of `http://faculty.etsu.edu/gardnerr/4217/notes/3-1.pdf`). This idea of intersecting a given set with open sets from a larger space is the inspiration for the ideas of this section.

Definition. Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, then the set $\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$ is a topology on Y called the *subspace topology.* With this topology, Y is called a *subspace* of X.

Note. The previous definition claims that $\mathcal{T}_Y$ is a topology. This is easy to confirm. First, $\varnothing, Y \in \mathcal{T}_Y$ since $\varnothing = Y \cap \varnothing$ and $Y = Y \cap X$. For finite intersections, we have

$$(U_1 \cap Y) \cap (U_2 \cap Y) \cap \cdots \cap (U_n \cap Y) = (U_1 \cap U_2 \cap \cdots \cap U_n) \cap Y \in \mathcal{T}_Y$$

and for arbitrary unions

$$\cup_{\alpha \in J}(U_\alpha \cap Y) = (\cup_{\alpha \in J} U_\alpha) \cap Y \in \mathcal{T}_Y.$$

Now we give a method by which to find a basis for the subspace topology.

Lemma 16.1. If $\mathcal{B}$ is a basis for the topology of X then the set $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.

Definition. If Y is a subspace of X, then a set U is *open in* Y (or *open relative to* Y) if U is in the (subspace) topology of Y. Set U is *open in* X if it belongs to the topology of X.

Note. It is possible for a set U to be open in Y but not open in X. Let $X = \mathbb{R}$ and $Y = [0, 2]$ where $\mathbb{R}$ has the standard topology and Y has the subspace topology. Then $U = [0, 1)$ is open in Y since $U = (-1, 1) \cap [0, 2]$ where $(-1, 1)$ if open in $\mathbb{R}$, but U is not open in $\mathbb{R}$. The following result gives a condition under which open sets in the subspace topology are also open in the "superspace" topology.

Lemma 16.2. Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Note. We now explore the interplay between the subspace topology and the order topology and product topology.

Lemma 16.3. If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Note. For Y as a subspace of X where X has a simple order relation on it (which Y will inherit), then the order topology on Y may or may not be the same as the subspace topology on Y, as illustrated in the following examples.

Example 1. Let $X = \mathbb{R}$ with the order topology (which for $\mathbb{R}$ is the same as the standard topology) and let $Y = [0, 1]$ have the subspace topology. A basis for the order topology on $\mathbb{R}$ is $\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}, a < b\}$ (by the definition of "order topology," since $\mathbb{R}$ has neither a least or greatest element) and so a basis for the subspace topology is the set of all sets of the form $(a, b) \cap Y$ (by the definition of "subspace topology"). The subspace basis is then sets of the following forms: (a, b), $[0, b)$, $(a, 1]$, Y, or $\varnothing$ (where $0 \leq a < b \leq 1$). Each of these sets is also a basis element for the order topology on Y (in this case, Y has a least and greatest element), and conversely. So the basis for the subspace topology is the same as the basis for the order topology.

Example 2. Let $X = \mathbb{R}$ with the order topology and let $Y = [0, 1) \cup \{2\}$. In the subspace topology on Y, then singleton $\{2\}$ is open: $\{2\} = (3/2, 5/2) \cap Y$. But in the order topology on Y, $\{2\}$ is not open since a basis element for the order topology on Y that contains 2 is of the form $\{x \in Y \mid a < x \leq 2, a \in Y\}$. But such an $a \in Y$ $(a \neq 2)$ must be in $[0, 1)$ and so cannot be a subset of $\{2\}$. So the subspace topology on Y is different from the order topology on Y.

Example 3. Let $X = \mathbb{R}^2$ with the dictionary order and the order topology, and let $Y = [0,1]\times[0,1]$. With th eorder topology on Y, the space is calls the *ordered square* (where we denote $I = [0,1]$ and $I_0^2 = Y$). Consider the set $A = \{1/2\} \times (1/2, 1]$. This set is open in Y under the subspace topology (see Figure 16.1(a) on page 91; $A = Y \cap (\{1/2\} \times (1/2, 3/2))$), but A is not open unser the order topology since an open set containing the point $(1/2, 1)$ must also contain other points in Y (points to the right of the vertical line $x = 1/2$; see Figure 16.1(b) on page 91).

Note. Given the previous two examples, it is natural to ask: "Under what conditions will the subspace topology and order topology be the same?" The following definition addresses this (at least in part).

Definition. Given an ordered set X, a subset $Y \subset X$ is *convex* in X if for each pair of points $a, b \in Y$ with $a < b$, the entire interval (a, b) lies in Y.

Note. Both intervals and rays are convex sets. Notice that neither Example 2 nor Example 3 involve convex sets.

Theorem 16.4. Let X be an ordered set in the order topology. Let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the subspace topology on Y.

Note. Since there is potentially ambiguity when putting a topology on a subset Y of an ordered set X with the order topology, Munkres states that "*we shall assume that* Y *is given the subspace topology* unless we specifically state otherwise" [Munkres' emphasis].

Revised: 5/30/2016