Section 17. Closed Sets and Limit Points

Note. In this section, we finally define a "closed set." We also introduce several traditional topological concepts, such as limit points and closure.

Definition. A subset A of a topological space X is *closed* if set $X \setminus A$ is open.

Note. Both $\varnothing$ and X are closed.

Example 1. The subset $[a, b]$ if $\mathbb{R}$ under the standard topology is closed because its compliment $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, +\infty)$ is open. Also, $[a, +\infty)$ and $(-\infty, a]$ are closed, so that the terms "closed interval" and "closed ray" are justified.

Example 5. Let $Y = [0, 1] \times (2, 3) \subset \mathbb{R}$ is the subspace topology (with $\mathbb{R}$ under the standard topology). Then both $[0, 1]$ and $(2, 3)$ are open since $[0, 1] = (-1/2, 3/2) \cap Y$ and $(2, 3) = (2, 3) \cap Y$. Since, in topological space Y, $[0, 1]$ has $(2, 3)$ as its compliment and $(2, 3)$ has $[0, 1]$ as its compliment, then $[0, 1]$ and $(2, 3)$ are also both closed. We'll encounter this type of example again when we address connectedness in Chapter 3.

Note. The following result should be familiar to you from the setting of Analysis 1; see Theorem 3-3 of `http://faculty.etsu.edu/gardnerr/4217/notes/3-1.pdf`.

Theorem 17.1. Let X be a topological space. Then the following conditions hold:

(1) $\varnothing$ and X are closed.

(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed

Note. Munkres takes the following definition of "closed relative to $Y \subset X$" which seems to be a little inconsistent with the definition of "open relative to $Y \subset X$," but this is resolved in the theorem following the definition.

Definition. If Y is a subspace of X, we say that a set A is *closed in* Y (or *closed relative to* Y) if $A \subset A$ is closed in the subspace topology of Y (that is, if $Y \setminus A$ is open in the subspace topology of Y).

Theorem 17.2. Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Note. The following result is analogous to Lemma 16.2, but is for closed sets instead of pen sets. The proof is left as Exercise 17.2.

Theorem 17.3. Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

Definition. Given a subset A of a topological space X, the *interior* of A, denoted $\text{Int}(A)$, is the union of all open subsets contained in A. The *closure* of A, denoted $\overline{A}$ (or sometimes $\text{Cl}(A)$) is the intersection of all closed sets containing A.

Note. The interior of A is open by part (2) of the definition of topology. The closure of A is closed by part (2) of Theorem 17.1. Of course, $\text{Int}(A) \subset A \subset \overline{A}$.

Note. You may have the concept of an interior point to a set of real numbers in your Analysis 1 class (see page 7 of `http://faculty.etsu.edu/gardnerr/4217/notes/3-1.pdf`) where such points are defined using an ε definition. However, we do not (necessarily) have a way to measure distance in a topological space, so we cannot take this approach. Things are about to get much stranger when we define a "limit point" of a set using only the topology and not using a distance-based concept of "closeness." Once we consider metric spaces in Sections 20 and 21 we will be back to a setting similar to your Analysis 1 experience.

Lemma 17.A. Let A be a subset of topological space X. Then A is open if and only if $A = \text{Int}(A)$. A is closed if and only if $A = \overline{A}$.

Note. If Y is a subspace of X and $A \subset Y$, then A may have different closures in X and Y. So the symbol "$\overline{A}$" is ambiguous in this setting. Munkres reserves the symbol for the closure of A in X. The following result relates the closure of A in Y to the closure of A in X.

Theorem 17.4. Let Y be a subspace of X. Let $A \subset Y$ and denote the closure of A in X as $\overline{A}$. Then the closure of A in Y equals $\overline{A} \cap Y$.

Definition. A set A *intersects* a set B if $A \cap B \neq \varnothing$. An open set $U \subset X$ is a *neighborhood* of $x \in X$ if $x \in U$.

Theorem 17.5 Let A be a subset of the topological space X.

(a) Then $x \in \overline{A}$ if and only if every neighborhood of x intersects A.

(b) Supposing the topology of X is given a basis , then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

Example 6. Let $X = \mathbb{R}$ under the standard topology. If $A = (0, 1]$ then $\overline{A} = [0, 1]$. If $B = \{1/n \mid n \in \mathbb{N}\}$ then $\overline{B} = \{0\} \cup B$. If $C = \{0\} \cup (1, 2)$ then $\overline{C} = \{0\} \cup [1, 2]$. $\underline{\mathbb{Q}} = \mathbb{R}$, $\overline{\mathbb{Z}} = \mathbb{Z}$, $\overline{\mathbb{R}_+} = \mathbb{R}_+ = \mathbb{R}_+ \cup \{0\}$ where $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$.

Note. Now we define a "limit point" of a set without an appeal to closeness, but only by using properties of the topology (which means we only use open sets).

Definition. If A is a subset of topological space X and if $x \in X$ then x is a *limit point* (or *cluster point* or *point of accumulation*) of A if every neighborhood of x intersects A in some point other than x itself.

Example. Consider the set $A = \{0\} \cup (1, 2]$ in $\mathbb{R}$ under the standard topology. Then $\overline{A} = \{0\} \cup [1, 2]$, $\text{int}(A) = (1, 2)$, and the limit points of A are the points in $[1, 2]$. So $0 \in A$ is a point of closure and a limit point but not an element of A, and the points in $(1, 2] \subset A$ are points of closure and limit points.

Note. The following result gives a relationship between the closure of a set and its limit points.

Theorem 17.6 Let A be a subset of the topological space X. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.

Corollary 17.7. A subset of a topological space is closed if and only if it contains all its limit points.

Note. We will study the "separation axioms" more in Section 31, but Munkres introduces one of the axioms at this stage (as opposed to "axioms," these are more restrictions on a topological space which guarantee the existence of enough open sets for certain points/sets to be separated by open sets). Consider the topological space $X = \{a, b, c\}$ with open set in $\mathcal{T} = \{\varnothing, \{a, b\}, \{b\}, \{b, c\}, X\}$ (see Figure 17.3 on page 98). Whereas $\mathbb{R}$ with the standard topology has every singleton as a closed set, this is not the case for topology $\mathcal{T}$ on X since $\{b\}$ is not closed (because $X \setminus \{b\} = \{a, c\}$ is not open) We give another definition and see that this topological space has another undesirable property.

Definition. Let $x_1, x_2, \ldots$ be a sequence of points in topological space X. The sequence *converges* to $x \in X$ if for every neighborhood U of x there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n \in U$. In this case, x is called a *limit* of the sequence.

Note. The above definition is similar to the definition from Calculus 2, but the "for all $\varepsilon > 0$" part of Calculus 2 is replaced with "for every neighborhood" here. In the topology on $X = \{a, b, c\}$ mentioned above, notice that the sequence $b, b, b, \ldots$ converges to b (because every neighborhood of b contains all elements of the sequence), converges to a (because every neighborhood of a, the smallest such neighborhood being $\{a, b\}$, contains all elements of the sequence), and converges to c (because every neighborhood of c, the smallest such neighborhood being $\{b, c\}$, contains all elements of the sequence). Munkres takes this nonuniqueness of limits of sequences and the fact that singletons may not be closed sets, as motivation for the following definition.

Definition. A topological space X is a Hausdorff space if for each pair of distinct points $x_1, x_2 \in X$, there exist neighborhoods U_1 of x_1 and U_2 of x_2 such that $U_1 \cap U_2 = \varnothing$.

Note. Theorems 17.8 and 17.10 show that the Hausdorff condition resolves the concerns expressed above.

Theorem 17.8. Every finite point set in a Hausdorff space X is closed. In particular, singletons form closed sets in a Hausdorff space.

Note. The following result introduces a new separation axiom. Notice that, by Theorem 17.8, Hausdorff spaces satisfy the new condition.

Theorem 17.9. Let X be a space satisfying the "T_1 Axiom" (namely, that all finite point sets are closed). Let A be a subset of X. Then x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Note. On page 99, Munkres comments that "...most of the spaces that are important to mathematicians are Hausdorff spaces. The following two theorems give some substance to these remarks."

Theorem 17.10. If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.

Theorem 17.11. Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.

Note. The proofs of these claims made in Theorem 17.11 are to be justified in Exercises 17.10, 17.11, and 17.12.

Revised: 6/2/2016