Section 18. Continuous Functions

Note. Continuity is *the* fundamental concept in topology! When you hear that "a coffee cup and a doughnut are topologically equivalent," this is really a claim about the existence of a certain continuous function (this idea is explored in depth in Chapter 12, "Classification of Surfaces"). We start by reviewing some continuity ideas from Analysis 1 (MATH 4217/5217).

Note. The standard definition of continuity of a real valued function of a real variable at a point x_0 in the domain of the function f, $\mathcal{D}(f)$, is as follows (see `http://faculty.etsu.edu/gardnerr/4217/notes/4-1.pdf`, page 2):

> **Definition.** Suppose f is a function and $x_0 \in \mathcal{D}(f)$. Then f is *continuous at point* x_0 if
>
> $$\text{for all } \varepsilon > 0 \text{ there exits } \delta(\varepsilon) > 0 \text{ such that}$$
>
> $$\text{for all } |x - x_0| < \delta(\varepsilon) \text{ and } x \in \mathcal{D}(f) \text{ we have } |f(x) - f(x_0)| < \varepsilon.$$

We then say that f is continuous a set $A \subset \mathbb{R}$ if f is continuous at each point of A.

Note. The following is a consequence of the previous definition (see Theorem 4-5 in the Analysis 1 notes mentioned above):

> **Theorem 4-5.** Let $f : X \to Y$. Then f is continuous on $\mathcal{D}(f)$ if and only if for all open sets $V \subset Y$, we have $f^{-1}(V)$ is open relative to $\mathcal{D}(f)$.

Inspired by this result, we have the following as our (standard) definition of continuous functions. We take functions defined on all of topological space X, so there is no need for the "open relative to" part of the above definition.

Definition. Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* if for each open subset V of Y, the set $f^{-1}(V)$ is open in X.

Note. In Calculus 1, continuity is defined based on limits (which are defined using ε's and δ's), however we have not yet defined the limit of a function (though we have defined limit points of sets). Our definition is based entirely on open sets and we should in fact state that f is continuous *relative* to the topologies on X and Y.

Lemma 18.A. Let $f : X \to Y$, let $\mathcal{B}$ be a basis for the topology on Y, and let $\mathcal{S}$ be a subbasis for the topology on Y.

(1) f is continuous if $f^{-1}(B)$ is open in X for each $B \in \mathcal{B}$.

(2) f is continuous if $f^{-1}(S)$ is open in X for each $X \in \mathcal{S}$.

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ (with $\mathbb{R}$ having the standard topology) by continuous (under our definition). Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. Then the interval $V = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ is open in the range space $\mathbb{R}$. Therefore, $f^{-1}(V)$ is open in the domain space $\mathbb{R}$. Now $x_0 \in f^{-1}(V)$ so there is some basis element (a, b) containing x_0 with $(a, b) \subset f^{-1}(V)$ (recall that the standard topology on $\mathbb{R}$ is *defined* as the topology with basis $\{(a, b) \mid a, b \in \mathbb{R}, a < b\}$; see page 81). Choose $\delta = \min\{f(x_0) - a, b - f(x_0)\}$. Then for $x \in (f(x_0) - \delta, f(x_0) + \delta)$ (that is, for $|x - f(x_0)| < \delta$) we have $f(x) \in V$ (that is, $|f(x) - f(x_0)| < \varepsilon$). So our definition of continuity implies the ε/δ definition of continuity. In fact, in the setting $f : \mathbb{R} \to \mathbb{R}$, the ε/δ definition of continuity implies the open set definition (so there are equivalent); you are asked to show this in Exercise 18.1.

Example 3. Let $\mathbb{R}$ have the standard topology and $\mathbb{R}_\ell$ have the lower limit topology. Let $f : \mathbb{R} \to \mathbb{R}_\ell$ be the identity function $f(x) = x$ (which is of course continuous when mapping $\mathbb{R} \to \mathbb{R}$). Then f is *not* continuous here since for $a < b$, $[a, b)$ is open in $\mathbb{R}_\ell$ for $f^{-1}([a, b)) = [a, b)$ is not open in $\mathbb{R}$.

Note. In the following theorem it is shown (in the (1)⇔(4)) that $f : X \to Y$ is continuous if and only if for each $x \in X$ and each neighborhood V of $f(x)$ there is a neighborhood U of x such that $f(U) \subset V$. So the "for all $\varepsilon > 0$" has been the "there exists $\delta > 0$" has been replaced with "there is a neighborhood U." The implication "$|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon$" is replaced with $x \in U \Rightarrow f(x) \in V$.

Theorem 18.1. Let X and Y be topological spaces. let $f : X \to Y$. Then the following are equivalent:

(1) f is continuous.

(2) For every subset Z of X, one has $f(\overline{A}) \subset \overline{f(A)}$.

(3) For every closed subset B of Y, the set $f^{-1}(B)$ is closed in X.

(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Note. Recall that, in a general sense, an *isomorphism* π between two mathematical objects is a one to one and onto mapping which "preserves structure." For example, in a group the structure is the binary operation, so we require $\pi(a*b) = \pi(a)*'\pi(b)$. In a graph, the structure is connectivity so that if vertices v and w are adjacent in G, then we require that $\pi(v)$ and $\pi(w)$ are adjacent in $\pi(G)$ (and conversely). In a vector space the structure is linear combination, so we require $a\vec{v_1} + \vec{v_2} = \vec{w}$, then $a\pi(\vec{v_1}) + b\pi(\vec{v_2}) = \vec{w}$. In a topological space, the structure is the collection of open sets. So we want U open in X to imply $\pi(U)$ open in Y (and conversely). The term "isomorphism" is not used in the topological setting, but the concept is the same. We have the following.

Definition. Let X and Y be topological spaces. Let $f : X \to Y$ be a bijection (one to one and onto). If both f and $f^{-1} : Y \to X$ are continuous, then f is a *homeomorphism*.

Note. If $f : X \to Y$ is a homeomorphism then U is open in X if and only if $f(U)$ is open in Y. Any property in X that is expressed entirely in terms of the topology on X yields through the homeomorphism the corresponding property in Y. Such a property is called a *topological property* of X. Examples of such properties are open/closed, limit points of a set, limits of a sequence, a basis or subbasis for the topology, and (as we will see in Chapter 3) connectedness and compactness.

Definition. Let $f : X \to Y$ be an injective (one to one) continuous map. Let $Z = f(X)$ (so that f is onto Z) be considered a subspace of Y. Let $f' : X \to Z$ be the restriction of f to Z (so f' is a bijection). If f' is a homeomorphism of X with Z, then $f : X \to Y$ is a *topological imbedding* (or simply *imbedding*) of X in Y.

Example 5. Consider $F : (-1, 1) \to \mathbb{R}$ defined by $F(x) = x/(1 - x^2)$. Then F is continuous and one to one (since $F'(x) = (1 + x^2)/(1 - x^2)^2 \geq 0$) and is continuous on $\mathbb{R}$. So F is a homeomorphism (where $\mathbb{R}$ has the standard topology and $(-1, 1)$ has the subspace topology).

Example 6. Here we give an example of a continuous bijective function which has an inverse which is not continuous. Consider $[0, 1)$ with the subspace topology (as a subspace of $\mathbb{R}$ with the standard topology) and $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ with the subspace topology (as a subspace of $\mathbb{R}^2$ with the standard topology). Define $f : [0, 1) \to S^1$ as $f(t) = (\cos 2\pi t, \sin 2\pi t)$. Then f is a continuous bijection but f^{-1} is not continuous since $f([0, 1/2)) = \{(\cos 2\pi t, \sin 2\pi t) \mid t \in [0, 1/2)\}$ and so the inverse image (under mapping f; that is, $(f^{-1})^{-1}([0, 1/2)) = f([0, 1/2)))$ of open set $[0, 1/2)$ is not open.

Theorem 18.2. Rules for Constructing Continuous Functions.

Let X, Y, and Z be topological spaces.

(a) (Constant Function) If $f : X \to Y$ maps all of X into a single point $y_0 \in Y$, then f is continuous.

(b) (Inclusion) if A is a subspace of X, the inclusion function $j : A \to X$ is continuous.

(c) (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.

(d) (Restricting the Domain) If $f : X \to Y$ is continuous and if A is a subspace of X, then the restricted function $f|_A : A \to Y$ is continuous.

(e) (Restricting or Expanding the Range) let $f : X \to Y$ be continuous. If X is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the functions $h : X \to Z$ obtained by expanding the range of f is continuous.

(e) (Local Formulation of Continuity) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|_{U_\alpha}$ is continuous for each α.

Note. The following result, which is fairly easy to prove, will be extremely useful in the future.

Theorem 18.3. The Pasting Lemma for Closed Sets.
Let $X = A \cup B$ where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cup B$, then f and g combine (or "paste") to give a continuous function $h : X \to Y$ defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

Note. As you can see in the proof of the previous theorem, we can replace "closed" with "open" to get the following.

Corollary 18.A. The Pasting Lemma for Open Sets.
Let $X = A \cup B$ where A and B are open in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cup B$, then f and g combine (or "paste") to give a continuous function $h : X \to Y$ defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

Note. Now we define a "limit point" of a set without an appeal to closeness, but only by using properties of the topology (which means we only use open sets).

Note. Munkres gives examples on page 109 of applications of The Pasting Lemma involving piecewise defined functions on $\mathbb{R}$.

Theorem 18.4. Maps into Products.

Let $f : A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$ where $f_1 : A \to X$ and $f_2 : Y \to B$. Then f is continuous if and only if the functions f_1 and f_2 are continuous.

Definition. The functions $f_1 : A \to X$ and $f_2 : A \to Y$, where $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$ for $f : A \to X \times Y$ above, are called *coordinate functions* of f.

Note. Munkres states (see page 110): "There is no useful criterion for the continuity of a map $f : A \times B \to X$ whose *domain* is a product space." For example, we might expect that if $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous if $f(x, y_0)$ and $f(x_0, y)$ are continuous for each $y_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ (that is, if f is continuous "in each variable separately") then $f(x, y)$ is continuous. This is not the case, though, as shown by

$$f(x,y) = \begin{cases} xy/(x^2+y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

This claim is justified in Exercise 18.12.

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