Section 19. The Product Topology

Note. In Section 15 we defined the product topology on the product of *two* topological spaces X and Y. In this section we consider arbitrary products of topological spaces and give two topologies on these spaces, the box topology and the product topology. For finite products, these topologies coincide.

Definition. Let J be an (arbitrary) index set. Given a set X, define a *J-tuple* of elements of X to be a function $\mathbf{x} : J \to X$. If α is an element of J, denote $\mathbf{x}((\alpha)$ as x_α called the αth *coordinate* of $\mathbf{x}$. We often denote $\mathbf{x}$ as $(x_\alpha)_{\alpha \in J}$ and denote the set of all J-tuples of elements of X as X^J.

Definition. Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; let $X = \cup_{\alpha \in J} A_\alpha$. The *Cartesian product* of this indexed family, denoted by $\prod_{\alpha \in J} A_\alpha$, is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, $\prod_{\alpha \in J} A_\alpha$ is the set of all functions $\mathbf{x} : J \to \cup_{\alpha \in J} A_\alpha$ such that $\mathbf{x}(\alpha) \in A_\alpha$ for each $\alpha \in J$.

Definition. Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. We take as a basis for a topology on the product space $\prod_{\alpha \in J} X_\alpha$ the collection of all sets of the form $\prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α for each $\alpha \in J$. The topology generated by this basis is called the *box topology*.

Note. When $J = \{1, 2\}$, the box topology is the same as the product topology on $X \times Y$ from Section 15.

Note. The collection of all sets of the form $\prod_{\alpha \in J} U_\alpha$ do, in fact, form a basis for a topology on $\prod_{\alpha \in J} X_\alpha$. First, $\prod_{\alpha \in J} X_\alpha$ itself is such a set so each $\mathbf{x} \in \prod_{\alpha \in J} X_\alpha$ is in some basis element. Second, suppose $\mathbf{x} \in \left(\prod_{\alpha \in J} U_\alpha\right) \cap \left(\prod_{\alpha \in J} V_\alpha\right) = \prod_{\alpha \in J} U_\alpha \cap V_\alpha$. Notice that each U_α, V_α is open in X_α, so $\prod_{\alpha \in J} U_\alpha \cap V_\alpha$ is a basis element and the definition of basis is satisfied.

Definition. Define the *product mapping* $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ for $\beta \in J$ as the function assigning to each element of $\prod_{\alpha \in J} X_\alpha$ its βth coordinate, $\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$. Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta\}$$

and let $\mathcal{S}$ denote the union of these collections, $\mathcal{S} = \cup_{\beta \in J} \mathcal{S}_\beta$. The topology generated by the subbasis $\mathcal{S}$ is the *product topology* and this topology on $\prod_{\alpha \in J} X_\alpha$ is the *product space.*

Note. In order to verify that $\mathcal{S}$ really is a subbasis for a topology, we must only show that there are sets in $\mathcal{S}$ which union to give $\prod_{\alpha \in J} X_\alpha$ (by the definition of subbasis). Now X_β is open in X_β and $\pi_\beta^{-1}(X_\beta) = \prod_{\alpha \in J} X_\alpha \in \mathcal{S}$ (see Figure 15.2 on page 88 for added motivation of this claim) so, in fact, $\mathcal{S}$ really is a subbasis for a topology.

Theorem 19.1. Comparison of the Box and Product Topologies. The box topology on $\prod_{\alpha \in J} X_\alpha$ has as a basis all sets of the form $\prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α for each $\alpha \in J$. The product topology on $\prod_{\alpha \in J} X_\alpha$ has as a basis all sets of the form $\prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α for each $\alpha \in J$ and $U_\alpha = X_\alpha$ except for finitely many values of α.

Note. Of course, if J is a finite set then the box topology and the product topology on $\prod_{\alpha \in J} X_\alpha$ coincide (since, by Theorem 19.1, they have bases with the same elements). Also, in general the box topology is a finer topology than the product topology on $\prod_{\alpha \in J} X_\alpha$.

Note. Because the product topology is coarser (or "weaker") than the box topology, more "desirable" properties hold in the product topology. This will be illustrated in several results. For this reason, when considering $\prod_{\alpha \in J} X_\alpha$ (which we abbreviate $\prod X_\alpha$) we assume that it has the product topology, unless stated otherwise.

Note. The following gives bases for the box and product topologies based on bases of the constituent topologies on the X_α's. It's proof is left as Exercise 19.1.

Theorem 19.2. Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form $\prod_{\alpha \in J} B_\alpha$ where $B_\alpha \in \mathcal{B}_\alpha$ for each $\alpha \in J$ serves as a basis for the box topology on $\prod X_\alpha$.

The collection of all sets of the same form where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all remaining indices serves as a basis for the product topology on $\prod X_\alpha$.

Example 1. Consider $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$. A basis for $\mathbb{R}$ (with the standard topology) consists of all open intervals in $\mathbb{R}$. So, by Theorem 19.2, a basis for the box topology (or product topology; these coincide here) is the set of all products of the form $(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$. This is the "standard topology" on $\mathbb{R}^n$. Notice that this example can be easily modified to find bases for the box topology and the product topology on $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots$ (see Exercise 19.7).

Note. The following three theorems give examples of properties that hold for both the box topologies and the product topologies. The proofs of Theorems 19.3 and 19.4 are left as exercises 19.2 and 19.3, respectively.

Theorem 19.3. Let A_α be a subspace of X_α for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology or if both products are given the product topology.

Theorem 19.4. If each space X_α is a Hausdorff space then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.

Theorem 19.5. Let $\{X_\alpha\}$ be an indexed family of spaces and let $A_\alpha \subset X_\alpha$ for each $\alpha \in J$. If $\prod X_\alpha$ is given either the product or the box topology then $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$.

Note. The previous three results give no evidence for a preference for the product topology over the box topology. The following theorem and example concerning continuous functions gives some evidence for the preference.

Theorem 19.6. Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given as $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha : A \to X_\alpha$ for each $\alpha \in J$. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each functions f_α is continuous.

Note. The proof of Theorem 19.6 uses explicit use of the form of a subbasis for the product topology. The following example shows (in the notation of Theorem 19.6) that each f_α can be continuous, without f being continuous in the box topology.

Example 2. Consider $\mathbb{R}^\omega = \prod_{n=1}^{n} \mathbb{R}$ where each $\mathbb{R}$ has the standard topology and $\mathbb{R}^\omega$ has the box topology. Define $f : \mathbb{R} \to \mathbb{R}^\omega$ by the equation $f(t) = (t, t, t, \ldots)$. Then $f_n(t) = t$ and so $f : \mathbb{R} \to \mathbb{R}$ is continuous for each $n \in \mathbb{N}$ (and so if we put the product topology on $\mathbb{R}^\omega$ then f would be continuous by Theorem 19.6). Consider the basis element for the box topology $B = (-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$. Notice that $f^{-1}(B) = (-1, 1) \cup (-1/2, 1/2) \cup (-1/3, 1/3) \cup \cdots = \{0\}$ (notice that $f((-r, r)) = \prod_{i=1}^{\infty}(-r, r)$ so $(-1, 1)$ is the inverse image of all elements of $\mathbb{R}^\omega$ with all components equal and in $(-1, 1)$, $(-1/2, 1/2)$ is the inverse image of all elements of $\mathbb{R}^\omega$ with all components equal and in $(-1/2, 1/2)$, etc.). But B is open in $\mathbb{R}^\omega$ and $\{0\}$ is not open in $\mathbb{R}$. So f is not continuous.

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