Section 20. The Metric Topology

Note. The topological concepts you encounter in Analysis 1 are based on the metric on $\mathbb{R}$ which gives the distance between x and y in $\mathbb{R}$ which gives the distance between x and y in $\mathbb{R}$ as $|x - y|$. More generally, any space with a metric on it can have a topology defined in terms of the metric (which is ultimately based on an ε definition of pen sets). This is done in our Complex Analysis 1 (MATH 5510) class (see the class notes for Chapter 2 of `http://faculty.etsu.edu/gardnerr/5510/notes.htm`). In this section we define a metric and use it to create the "metric topology" on a set.

Definition. A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ having the following properties:

(1) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) (The Triangle Inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

The nonnegative real number $d(x, y)$ is the *distance* between x and y. Set X together with metric d form a *metric space*. Define the set $B_d(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\}$, called the ε*-ball* centered at x (often simply denoted $B(x, \varepsilon)$).

Definition. If d is a metric on X then the collection of all ε-balls $B_d(x, \varepsilon)$ for $x \in X$ and $\varepsilon > 0$ is a basis for a topology on X, called the *metric topology* induced by d.

Lemma 20.A. Let $B_d(x, \varepsilon)$ be a ε-ball in a topological space with the metric topology and metric d. Let $y \in B_d(x, \varepsilon)$. Then there is $\delta > 0$ such that $B_d(y, \delta) \subset B_d(x, \varepsilon)$.

Proof. Define $\delta = \varepsilon - d(x, y)$. Then for $z \in B_d(y, \delta)$ we have $d(y, z) < \delta = \varepsilon - d(x, y)$ and so, by the Triangle Inequality, $d(x, z) \leq d(x, y) + d(y, z) < \varepsilon$. So $y \in B_d(x, \varepsilon)$ and $B_d(y, \delta) \subset B_d(x, \varepsilon)$. ∎

Note. As usual, we must verify that the collection of sets in the previous definition really satisfies the definition of basis of a topology. First, every element $x \in X$ is in a basis element, say $B(x, 1)$. Second, let B_1 and B_2 be two basis elements and let $y \in B_1 \cap B_2$. Then from Lemma 20.A, there are $\delta_1 > 0$ and $\delta_2 > 0$ with $B(y, \delta_1) \subset B_1$ and $B(y, \delta_2) \subset B_2$. With $\delta = \min\{\delta_1, \delta_2\}$ we then have $B(y, \delta) \subset B_1 \cap B_2$. Since $B(y, \delta)$ is a basis element then the second part of the definition of basis is satisfied.

Note. The following result is in line with the *definition* of open set in a metric space.

Lemma 20.B. A set U is open in the metric topology induced by metric d if and only if for each $y \in U$ there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.

Proof. Let $y \in U$. By the definition of "topology generated by a basis" (see page 78), U is open if and only if there is a basis element $B_d(x, \varepsilon)$ with $y \in B_d(x, \varepsilon) \subset U$. No by Lemma 20.A, there is $B_d(y, \delta)$ a basis element with $y \in B_d(y, \delta) \subset B_d(x, \varepsilon) \subset U$. ∎

Example 1. For set X, define metric

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

(this is in fact a metric). The topology reduces the discrete topology on X.

Definition. Let X be a topological space. X is said to be *metrizable* if there exists a metric d on a set X that induces the topology of X. A *metric space* is a metrizable space X with a specific metric d that gives the topology of X.

Note. In Section 34 a condition is given which insures that a topological space is metrizable in Urysohn's Metrization Theorem. However, the study of metric spaces is more a topic of analysis than of topology. In the remainder of this section, we consider some specific metric with particular attention payed to $\mathbb{R}^n$ and $\mathbb{R}^\omega$.

Definition. Let X be a metric space with metric d. A subset A of X is *bounded* if there is some number M such that $d(a_1, a_2) \leq M$ for every pair $a_1, a_2 \in A$. If A is bounded and nonempty, the *diameter* of A is $\text{diam}(A) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$.

Note. The next result gives a metric for which every set is bounded.

Theorem 20.1. Let X be a metric space with metric d. Define $\overline{d} : X \times X \to \mathbb{R}$ by $\overline{d}(x,y) = \min\{d(x,y), 1\}$. Then $\overline{d}$ is a metric that induces the same topology as d.

Note. We now define two specific metrics on $\mathbb{R}^n$ and in Theorem 20.3 show that they induce the same topology on $\mathbb{R}^n$.

Definition. Given $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define the *norm* of $\mathbf{x}$ as $\|\mathbf{x}\| = (x_1^2 + x_2^2 + \cdots + x_n)^{1/2}$. Define the *Euclidean metric* on $\mathbb{R}^n$ as

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = ((x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2)^{1/2}.$$

Define the *square metric* ρ as

$$\rho(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}.$$

Note. The proof that the Euclidean metric actually is a metric is given in Exercise 20.9.

Note. The square metric clearly satisfies the first two parts of the definition of metric. For the Triangle Inequality, notice that for any $x_i, y_i, z_i \in \mathbb{R}$ we have by the Triangle Inequality on $\mathbb{R}$, $|x_i - z_1| \leq |x_i - y_i| + |y_i - z_i|$. So for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ we have $|x_i - z_i| \leq \rho(\mathbf{x}, \mathbf{y}) + \rho(\mathbf{y}, \mathbf{z})$ for $i = 1, 2, \ldots, n$. Hence

$$\rho(\mathbf{x}, \mathbf{z}) = \max_{1 \leq i \leq n}\{|x_i - z_i|\} \leq \rho(\mathbf{x}, \mathbf{y}) + \rho(\mathbf{y}, \mathbf{z}),$$

and the Triangle Inequality holds for ρ. Therefore, ρ is a metric on $\mathbb{R}^n$.

Note. In fact, the topologies on $\mathbb{R}^n$ induced by the Euclidean metric and the square metric are the same as the product topology on $\mathbb{R}^n$. To establish this (in Theorem 20.3) we need a preliminary result.

Lemma 20.2. Let d and d' be two metrics on the set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ is and only if for such $x \in X$ and each $\varepsilon > 0$, there exists a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$.

Theorem 20.3. The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Note. The following definition allows us to put a metric topology on an arbitrary product of copies of $\mathbb{R}$.

Definition. Given an arbitrary index set J and points $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha \in J}$ in $\mathbb{R}^J$, define a metric $\overline{\rho}$ on $\mathbb{R}^J$ as

$$\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup_{\alpha \in J}\{\overline{d}(x_\alpha, y_\alpha)\}$$

where $\overline{d}(x_\alpha, y_\alpha) = \min\{|x_\alpha - y_\alpha|, 1\}$ is the standard bounded metric on $\mathbb{R}$ (see Theorem 20.1). Metric $\overline{\rho}$ is the *uniform metric* on $\mathbb{R}^J$ and the metric topology it induces is the *uniform topology*.

Note. We now confirm that $\overline{\rho}$ is in fact a metric. Since $\overline{d}$ is a metric by Theorem 20.2, then $\overline{\rho}$ "clearly" satisfies parts (1) and (2) of the definition of a metric. For any $\mathbf{x}, \mathbf{y}, \mathbf{x} \in \mathbb{R}^J$, we have $\overline{d}(x_\alpha, z_\alpha) \le \overline{d}(x_\alpha, y_\alpha) + \overline{d}(y_\alpha, z_\alpha)$ for all $\alpha \in J$ by the Triangle Inequality for $\overline{d}$. So

$$\overline{\rho}(\mathbf{x}, \mathbf{z}) = \sup_{\alpha \in J}\{\overline{d}(x_\alpha, z_\alpha)\} \le \sup_{\alpha \in J}\{\overline{d}(x_\alpha, y_\alpha) + \overline{d}(y_\alpha, z_\alpha)\}$$

$$\le \sup_{\alpha \in J}\{\overline{d}(x_\alpha, y_\alpha)\} + \sup_{\alpha \in J}\{\overline{d}(y_\alpha, z_\alpha)\} = \overline{\rho}(\mathbf{x}, \mathbf{y}) + \overline{\rho}(\mathbf{y}, \mathbf{z}),$$

the Triangle Inequality holds for $\overline{\rho}$, and $\overline{\rho}$ is a metric.

Theorem 20.4. The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology. These three topologies are all different if J is infinite.

Note. As shown in the following theorem, $\mathbb{R}^J$ is metrizable if J is countable and (in this case) $\mathbb{R}^J = \mathbb{R}^\omega = \mathbb{R}^\mathbb{N}$ has the product topology. Munkres claims (without proof) that $\mathbb{R}^\omega = \mathbb{R}^\mathbb{N}$ under a different topology is not metrizable and that $\mathbb{R}^J$ is not metrizable if J is uncountable.

Theorem 20.5. Let $\overline{d}(a,b) = \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If $\mathbf{x}$ and $\mathbf{y}$ are two points in $\mathbb{R}^\omega = \mathbb{R}^\mathbb{N}$, define

$$D(\mathbf{x},\mathbf{y}) = \sup_{i\in\mathbb{N}} \left\{ \frac{\overline{d}(x_i,y_i)}{i} \right\}.$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$. That is, $\mathbb{R}^\omega$ under the product topology is metrizable.

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