Section 21. The Metric Topology (Continued)

Note. In this section we give a number of results for metric spaces which are familar from calculus and real analysis. We also give a couple of examples of nonmetrizable spaces.

Note. The following theorem shows that the usual ε/δ definition of continuity is equivalent to our definition of continuity in terms of inverse images of open sets when we have a metric.

Theorem 21.1. Let $f : X \to Y$. let X and Y be metrizable with metrics d_X and d_Y, respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \;\Rightarrow\; d_Y(f(x), f(y)) < \varepsilon.$$

Note. Recall from senior level analysis that a closed set contains its limit points (see Corollary 3-6(a) at `http://faculty.etsu.edu/gardnerr/4217/notes/3-1.pdf`). The following result for topological spaces is related to this familiar property from $\mathbb{R}$.

Lemma 21.2. The Sequence Lemma.

Let X be a topological space. Let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$. If X is metrizable and $x \in \overline{A}$ then there is a sequence $\{x_n\} \subset A$ such that $\{x_n\} \to x$.

Note. We will see in Example 3 of Section 28 a set $A = S_\Omega$ in a space $X = \overline{S}_\Omega$ where A has a limit point $x = \Omega$, but there is no sequence of elements of A which converge to $x = \Omega$. We will be able to conclude that this space is not metrizable. This might strike you as surprising that there is a difference between a limit point of a set and a limit of a sequence of elements of the set (since $\mathbb{R}$ is metrizable, this does not happen in senior level analysis).

Note. Recall from senior level analysis that function f is continuous at point x if and only if every sequence $\{x_n\} \to x$ in the domain of f satisfies $\{f(x_n)\} \to f(x)$. See Theorem 4-12 and Corollary 4-12 (concerning one-sided limits) at `http://faculty.etsu.edu/gardnerr/4217/notes/4-2.pdf`.

Theorem 21.3. Let $f : X \to Y$. If f is continuous then for every convergent sequence $\{x_n\} \to x$ in X, the sequence $\{f(x_n)\} \to f(x)$ in Y. If X is metrizable and for any sequence $\{x_n\} \to x$ in X we have $\{f(x_n)\} \to f(x)$ in Y, then f is continuous.

Note. The hypotheses of metrizability in the proofs of the second parts of Lemma 21.2 and Theorem 21.3 can be weakened. In this direction, we have the following definitions.

Definition. A topological space X is said to have a *countable basis at the point* x if there is a countable collection $\{U_n\}_{n\in\mathbb{N}}$ of neighborhoods of x such that any neighborhood U of x contains at least one of the sets U_n. A space that has a countable basis at each of its points is said to satisfy the *first countability axiom.*

Note. In the proofs of the second parts of Lemma21.2 we can replace the open balls $B_d(x, 1/n)$ with the intersection of the first n elements of a countable basis at x, $B_n = U_1 \cap U_2 \cap \cdots \cap U_n$. So we can change the hypothesis from "metrizable" to "first countable" and the second parts of Lemma 21.2 and Theorem 21.3 still hold (with no modifications to the proof of Theorem 21.3, since it used the second part of Lemma 21.2).

Note. Every metrizable space is first countable, simply take as a countable basis at x the open balls $B_d(x, 1/n)$ for $n \in \mathbb{N}$. The countability axioms are considered in more detail in Section 30.

Note. The proof of the following is given in Exercise 21.13.

Lemma 21.4. The addition, subtraction, and multiplication operations are continuous from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, and the quotient operation is a continuous function from $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ into $\mathbb{R}$.

Theorem 21.5. If X is a topological space and if $f, g : X \to \mathbb{R}$ are continuous, then $f + g$, $f - g$, and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all $x \in X$ then f/g is continuous.

Note. We now define the uniform convergence of a sequence of functions. You might recall that the uniform limit of a sequence of Riemann integrable functions $\{f_n\}$ on $[a, b]$ is Riemann integrable on $[a, b]$ and

$$\lim_{n\to\infty} \left(\int_a^b f_n(x)\,dx \right) = \int_a^b \left(\lim_{n\to\infty} f_n(x) \right) dx.$$

See Theorem 8-3 of `http://faculty.etsu.edu/gardnerr/4217/notes/8-1.pdf`.

Definition. Let $f_n : X \to Y$ be a sequence of functions from set X to metric space Y. let d be the metric for Y. The sequence of functions $\{f_n\}$ *converges uniformly* to the function $f : X \to Y$ if given $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \varepsilon$ for all $n > N$ and for all $x \in X$.

Note. The "uniform" in uniform convergence is the fact that for a given $\varepsilon > 0$, the same $N \in \mathbb{N}$ "works" FOR ALL $x \in X$ (that is, the $N \in \mathbb{N}$ works uniformly across set X).

Note. We do not address integration here, but we do consider continuity. Recall from senior level analysis that the uniform limit of a sequence of continuous functions is continuous. See Theorem 8-2 of `http://faculty.etsu.edu/gardnerr/4217/notes/8-1.pdf`.

Theorem 21.6. Uniform Limit Theorem.
Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If $\{f_n\}$ converges uniformly to f, then f is continuous.

Note. We now show two topological spaces are not metrizable by showing that the Sequence Lemma (Lemma 21.2, part 2) does not hold in the spaces. In fact, as commented after Theorem 21.3, this will also show that neither space is first countable.

Example 1. We claim that $\mathbb{R}^\omega = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ for all } i \in \mathbb{N}\}$ is not metrizable. We will show that $\mathbb{R}^\omega$ is not metrizable by the contrapositive of part 2 of the Sequence Lemma. Let A be the subset of $\mathbb{R}^\omega$ consisting of all points whose coordinates are positive:

$$A = \{(x_1, x_2, \ldots) \mid x_i > 0 \text{ for all } i \in \mathbb{N}\}.$$

Let $\mathbf{0} = (0, 0, \ldots)$. In the box topology, $\mathbf{0}$ belongs to $\overline{A}$, for if $B = (a_1, b_1) \times (a_2, b_2) \times \cdots$ is any basis element of the box topology containing $\mathbf{0}$ then each $b_i > 0$ and so $B \cap A \neq \varnothing$. Now let $\{\mathbf{a}_n\}$ be any sequence of elements in A, say $\mathbf{a}_n = (x_{1n}, x_{2n}, \ldots, x_{in}, \ldots)$. Then each $x_{in} > 0$ since $\mathbf{a}_n \in A$. Let $B' = (-x_{11}, x_{11}) \times$

$(-x_{22}, x_{22}) \times \cdots$. Then B' is a basis element for the box topology and $\mathbf{0} \in B'$. However, the ith component of $\mathbf{a}_n$ is not in the ith interval of B': $x_{in} \notin (-x_{in}, x_{in})$. So $\mathbf{a}_n \notin B'$ for all $n \in \mathbb{N}$. So B' is an open set in the box topology containing $\mathbf{0}$ which contains no element of $\{\mathbf{a}_n\}$. Therefore no sequence $\{\mathbf{a}_n\} \subset A$ can converge to $\mathbf{0}$. So by the Sequence Lemma (the contrapositive of the second part), $\mathbb{R}^\omega$ under the box topology in not metrizable.

Example 2. We claim that an uncountable product of $\mathbb{R}$ with itself under the product topology is not metrizable. Let J be an uncountable index set. As in the previous example, we show that $\mathbb{R}^J$ is not metrizable by the contrapositive of part 2 of the Sequence Lemma. Let A be the subset of $\mathbb{R}^J$ consisting of all points $\mathbf{x} = (x_1, x_2, \ldots)$ such that $x_\alpha = 1$ for all but finitely many values of α. Let $\prod U_\alpha$ be a basis element for the product topology which contains $\mathbf{0}$. So $U_\alpha = \mathbb{R}$ for all but finitely many values of α; see Theorem 19.1. Say $U_\alpha \neq \mathbb{R}$ for $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$. Let $\mathbf{x}' = (x', x_2', \ldots)$ where $x_\alpha' = 0$ for $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$ and $x_\alpha' = 1$ otherwise. Then $\mathbf{x}' \in A \cap \prod U_\alpha$. So any basis element for the product topology which contains $\mathbf{0}$ also contains some $\mathbf{x}' \in A$. Therefore, $\mathbf{0} \in \overline{A}$.

Now let $\{\mathbf{a}_n\}$ be any sequence of element in A. For a given $n \in \mathbb{N}$, let J_n denote the subset of J consisting of those (finite number of) indices α for which the αth coordinate of $\mathbf{a}_n$ is not 1. The union of all the sets J_n, $\cup_{n\in\mathbb{N}} J_n$, is a countable union of finite sets and so is countable. Because J is uncountable, then there is some index $\beta \in J$ where $\beta \notin \cup_{n\in\mathbb{N}} J_n$. So for each $\mathbf{a}_n$ in $\{\mathbf{a}_n\}$ we must have that the βth coordinate of $\mathbf{a}_n$ is 1.

Now let U_β be the open interval $(-1, 1)$ in $\mathbb{R}$ and let U be the open set $\pi_\beta^{-1}(U_\beta)$

in $\mathbb{R}^J$ where, in general, $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$. So $U = \pi_\beta^{-1}(U_\beta) - \prod_{\alpha \in J} X_\alpha$ where $X_\alpha = \mathbb{R}$ for $\alpha \neq \beta$ and $X_\beta = (-1, 1)$. Then U is a basis element for the product topology and this is why it is open. Also, $\mathbf{0} \in U$. However, no element of $\{\mathbf{a}_n\}$ is in U since the βth coordinate of all $\mathbf{a}_n$'s is 1. Therefore the sequence $\{\mathbf{a}_n\} \subset A$ can not converge to $\mathbf{0}$. So by the Sequence Lemma (the contrapositive of the second part), $\mathbb{R}^J$ under the product topology is not metrizable.

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