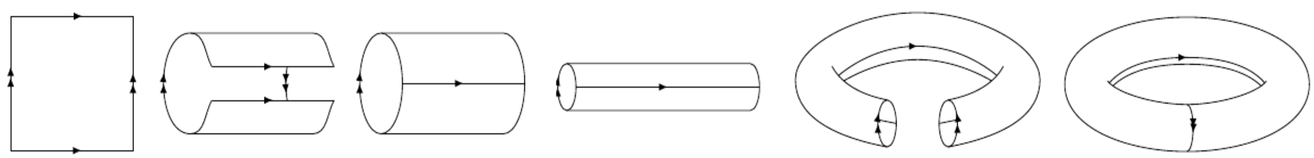
Section 22. The Quotient Topology

Note. In this section, we develop a technique that will later allow us a way to visualize certain spaces which cannot be embedded in three dimensions. The idea is to take a piece of a given space and glue parts of the border together. For example, if we connect the top and bottom of a rectangle and then connect the sides, then we produce a torus:



This image from `http://i.stack.imgur.com/FJaFe.png`.

Definition. Let X and Y be topological spaces. Let $p : X \to Y$ be a surjective (onto) map. The map p is a *quotient map* provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Note. If $p : X \to Y$ is continuous and surjective, it still may not be a quotient map. It might map an open set to a non-open set, for example, as we'll see below. However, if p is a quotient map then a subset $A \subseteq Y$ is closed if and only if $p^{-1}(A)$ is closed. This follows from the fact that $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$.

Definition. Let X and Y be topological spaces. Let $p : X \to Y$ be a surjective map. Set $C \subseteq X$ is *saturated* with respect to p if for all $y \in T$ such that $p^{-1}(\{y\}) \cap C \neq \varnothing$ we have $p^{-1}(\{y\}) \subseteq C$.

Note. If C is saturated with respect to p, then for some $A \subseteq Y$ we have $p^{-1}(A) = C$.

Lemma 22.A. Let X and Y be topological spaces. Then $p : X \to Y$ is a quotient map if and only if p is continuous and maps saturated open sets of X to open sets of Y.

Definition. The map : $f : X \to Y$ is an *open map* if for each open set $U \subseteq X$ the set $f(U)$ is open in Y. The map $f : X \to Y$ is a *closed map* if for each closed set $A \subseteq X$ the set $f(A)$ is closed in Y.

Note. If $p : X \to Y$ is continuous and surjective and p is either open or closed, then p is a quotient map. However, there are quotient maps that are neither open nor closed (see Munkres Exercise 22.3).

Example 22.1. Let $X = [0, 1] \cup [2, 3]$ be a subspace of $\mathbb{R}$ and let $Y = [0, 2]$ be a subspace of $\mathbb{R}$. Define $p : X \to Y$ as

$$p(x) = \begin{cases} x & \text{if } x \in [0,1] \\ x - 1 & \text{if } x \in [2,3]. \end{cases}$$

Then p is surjective and continuous. p is also closed (if set A contains limit points then set $f(A)$ contains its limit points). However, p is not open since the open set $[0, 1]$ of X is mapped to $[0, 1]$ which is not open in $Y = [0, 2]$.

Example 22.2. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection onto the first coordinate. Then π_1 is continuous and surjective. For any open set $\mathcal{O} \subseteq \mathbb{R} \times \mathbb{R}$, $\mathcal{O}$ is the countable union of basis elements of the form $U \times V$. Then $\pi_1(U \times V) = U$ is open in $\mathbb{R}$ and so π_1 carries open sets of $\mathbb{R} \times \mathbb{R}$ to open sets of $\mathbb{R}$. That is, π_1 is an open map. However, π_1 is not a closed map. The subset $C = \{(x, y) \mid xy = 1\}$ of $\mathbb{R} \times \mathbb{R}$ is closed in $\mathbb{R} \times \mathbb{R}$, but $\pi_1(C) = \mathbb{R} \setminus \{0\}$ which is not closed in $\mathbb{R}$.

Definition. If X is a space, A is a set, and $p : X \to A$ is surjective (onto) map, then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map. This topology is called the *quotient topology* induced by p.

Note. The previous definition claims the existence of a topology. This topology is simply the collection of all subsets of set A where $p^{-1}(A)$ is open in X. This is, in fact, a topology since $p^{-1}(\varnothing) = \varnothing$, $p^{-1}(A) = X$, $p^{-1}(\cup_{\alpha \in J} A_\alpha) = \cup_{\alpha \in J} p^{-1}(U_\alpha)$ where J is an arbitrary set, and $p^{-1}(\cap_{i=1}^n U_i) = \cap_{i=1}^n p^{-1}(U_i)$ (so each of the parts of the definition of topology are satisfied). The topology is unique since any additional or less open sets in set A would mean that p is not (by definition) a quotient map.

Example 22.3. Let p be the map of $\mathbb{R}$ onto $A = \{a, b, c\}$ given by

$$p(x) = \begin{cases} a & \text{if } x > 0 \\ b & \text{if } x < 0 \\ c & \text{if } x = 0. \end{cases}$$

Then the quotient topology on A must be $\mathcal{T} = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$.

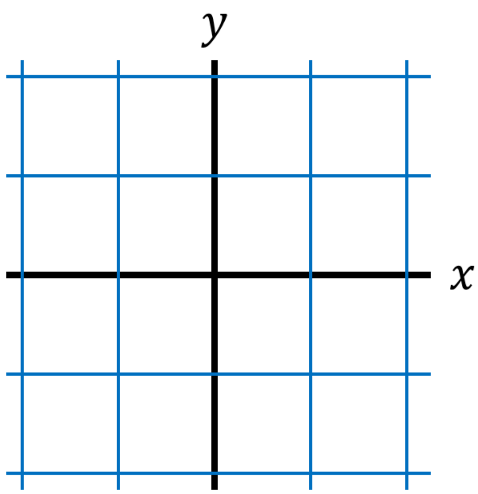
Note. The following idea introduces the technique by which we'll "cut and paste" a new space out of a given space.

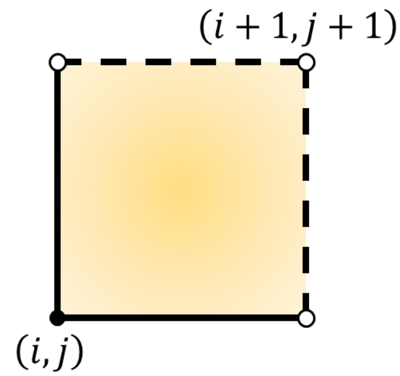
Definition. Let X be a topological space and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective (onto) map that carries each point of X to the element of X^* containing it. p is called the *projection map* from X to X^*. In the quotient topology on X^* induced by p, the space S^* under this topology is the *quotient space* of X.

Note. Recall that we have a partition of a set if and only if we have an equivalence relation on the set (this is Fraleigh's Theorem 0.22). So Munkres' approach in terms of partitions can be replaced with an approach based on equivalence relations. The idea of the quotient space is that points of the subsets in the partition (or the equivalent points under the equivalence relation) are "identified" with each other. For this reason, quotient spaces are sometimes called "identifying spaces" or "decomposition spaces." You will notice the parallel between quotient *spaces* and quotient *groups* in which all elements of a coset are identified.

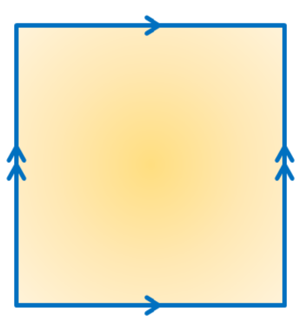
Note. In the following three examples, the winking smiley faces represent sets which are equivalent under the equivalence relation induced by the partition X^* of X. Each of the smiley faces in X is mapped by p to the same place in X^*.

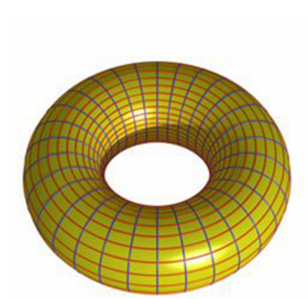
Example (Torus). Define an equivalence relation on $X = \mathbb{R} \times \mathbb{R}$ as $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 - x_2 \in \mathbb{Z}$ and $y_1 - y_2 \in \mathbb{Z}$. We partition X as follows:

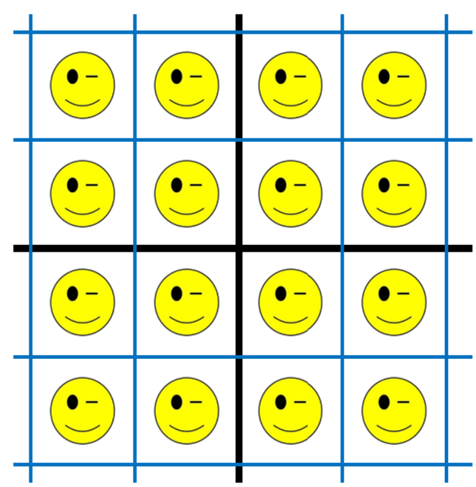




Then each 1×1 square contains exactly one representative from each equivalence class. Because of the correspondence of boundary points of a 1×1 square, the quotient space X^* is homeomorphic to the torus. The winking smiley faces represent sets equivalent under $\sim$.







The image of the torus is from `http://koc.wikia.com/wiki/File:Torus.jpg`.

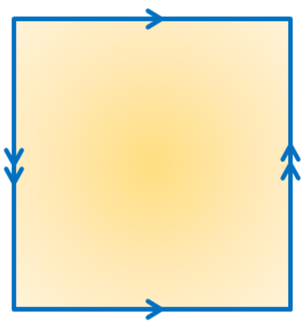
Example (Klein Bottle). Define an equivalence relation on $X = \mathbb{R} \times \mathbb{R}$ as $(x_1, y_1) \sim (x_2, y_2)$ if and only if

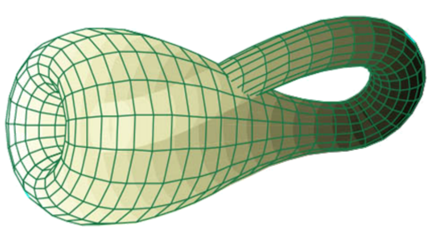
$$y_1 - y_2 \in \mathbb{Z} \text{ and } x_1 - x_2 \in 2\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$$

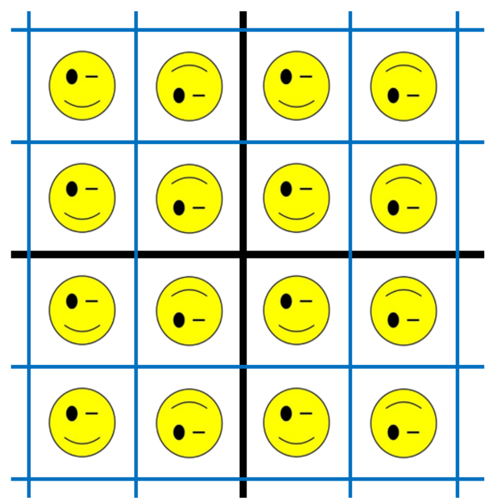
or

$$y_1 + y_2 \in \mathbb{Z} \text{ and } x_1 - x_2 \in \mathbb{Z} \setminus 2\mathbb{Z} = \{\ldots, -3, -1, 1, 3, 5, \ldots\}.$$

As in the previous example, a 1×1 square contains exactly one representative from each equivalence class. The quotient space X^* is homeomorphic to the *Klein bottle*.







The image of the Klein bottle is from `http://www.ediblegeography.com/wp-content/uploads/2013/03/Klein-bottle-460.jpg`

Example (Projective Plane). Define an equivalence relation on $X = \mathbb{R} \times \mathbb{R}$ as $(x_1, y_1) \sim (x_2, y_2)$ if and only if

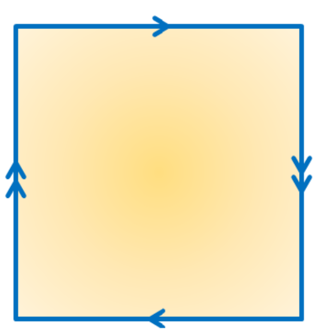
$$x_1 - x_2 \in 2\mathbb{Z} \text{ and } y_1 - y_2 \in 2\mathbb{Z}, \text{ or}$$

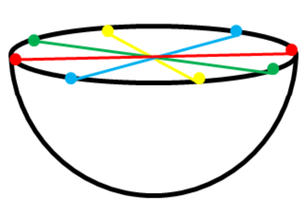
$$y_1 + y_2 \in \mathbb{Z},\ \lfloor y_1 \rfloor - \lfloor y_2 \rfloor \in 2\mathbb{Z},\ \text{and } x_1 - x_2 \in \mathbb{Z} \setminus 2\mathbb{Z},\ \text{or}$$

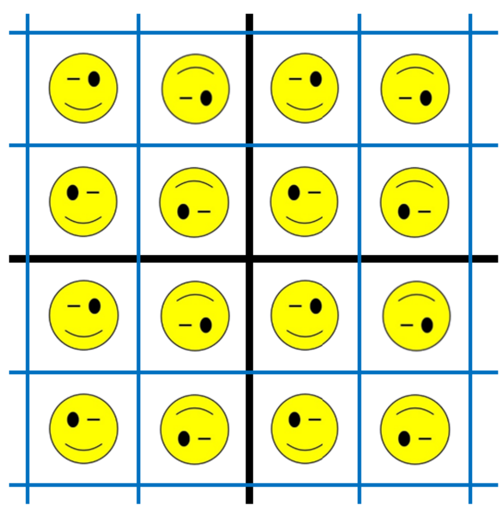
$$x_1 + x_2 \in \mathbb{Z},\ \lfloor x_1 \rfloor - \lfloor x_2 \rfloor \in 2\mathbb{Z},\ \text{and } y_1 - y_2 \in \mathbb{Z} \setminus 2\mathbb{Z},\ \text{or}$$

$$y_1 + y_2 \in \mathbb{Z},\ \lfloor y_1 \rfloor - \lfloor y_2 \rfloor \in \mathbb{Z} \setminus 2\mathbb{Z}, x_1 + x_2 \in \mathbb{Z},\ \text{and } \lfloor x_1 \rfloor - \lfloor x_2 \rfloor \in \mathbb{Z} \setminus 2\mathbb{Z}.$$

In terms of the boundary of the unit square, points "opposite the center" of the square are identified. As in the previous example, a 1×1 square contains exactly one representative from each equivalence class. The quotient space X^* is homeomorphic to the *projective plane*. An alternate model of the projective plane is given by taking a closed hemisphere (as a surface) and identifying points on the boundary which are "opposite the center" (or *antipodal* boundary points). We'll explore this approach again in the notes for Munkres' Section 60







Theorem 22.1. Let $p : X \to Y$ be a quotient map. Let A be a subspace of X that is saturated with respect to p. Let $q : A \to p(A)$ be the map obtained by restricting p to S, $q = p|_A$.

(1) If A is either open or closed in X, then a is a quotient map.

(2) If p is either an open or a closed map, then q is a quotient map.

Theorem 22.2. Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, for $y \in Y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous. f is a quotient map if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Corollary 22.3. Let $g : X \to Z$ be a surjective continuous map. Let X^* be the following collection of subsets of X: $X^* = \{g^{-1}\{z\}) \mid z \in Z\}$. Let X^* have the quotient topology.

(a) The map g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.

$$
\begin{tikzcd}
X \arrow[d, "p"'] \arrow[dr, "g"] & \\
X^* \arrow[r, "f"'] & Z
\end{tikzcd}
$$

(b) If Z is Hausdorff, so is X^*.

Revised: 6/6/2015