Supplement to Section 22. Topological Groups

Note. In this supplement, we define topological groups, give some examples, and prove some of the exercises from Munkres.

Definition. A *topological group* G is a group that is also a topological space satisfying the T_1 Axiom (that is, there is a topology on the elements of G and all sets of a finite number of elements are closed) such that the map of $G \times G$ into G defined as $(x, y) \mapsto x \cdot y$ (where $\cdot$ is the binary operation in G) and the map of G into G defined as $x \mapsto x^{-1}$ (where x^{-1} is the inverse of x in group G; this mapping is called *inversion*) are both continuous maps.

Note. Recall that a Hausdorff space is T_1 by Theorem 17.8. It is common to define a topological group to be one with a Hausdorff topology as opposed to simply T_1. For example, see Section 22.1, "Topological Groups: The General Linear Groups," in Royden and Fitzpatrick's *Real Analysis* 4th edition (Prentice Hall, 2010) and my online notes: `http://faculty.etsu.edu/gardnerr/5210/notes/22-1.pdf`.

Note. When we say the map of $G \times G$ into G defined as $(x, y) \mapsto x \cdot y$ is continuous, we use the product topology on $G \times G$. Since inversion maps G to G, the topology on G is used both in the domain and codomain.

Note. Every group G is a topological group. We just equip G with the discrete topology. Continuity of the binary operation follows at $(x, y) \in G \times G$ by taking the open set $\mathcal{O} = \{x \cdot y\}$ (the "δ set") for any given open set in $G \times G$ containing (x, y) (the "ε set").

Example. (Exercise 22S.2(b)) The additive group of real numbers $\mathbb{R}$ (under the usual topology) is a topological group. To establish this, we first show that addition is continuous. Let $x, y) \in \mathbb{R} \times \mathbb{R}$ and let $\varepsilon > 0$. Let $\delta = \varepsilon/2$. Consider the open set $\mathcal{O} = \{x' \mid |x - x'| < \delta\} \times \{y' \mid |y - y'| < \delta\}$ in $\mathbb{R} \times \mathbb{R}$. For all $(x', y') \in \mathcal{O}$ we have

$$\begin{aligned} |(x+y) - (x'+y')| &= |(x - x') + (y - y')| \\ &\le |x - x'| + |y - y'| \text{ by the Triangle Inequality} \\ &\qquad\qquad\qquad\qquad \text{for absolute value} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

So addition is continuous at (x, y) and since (x, y) is an arbitrary point of $\mathbb{R} \times \mathbb{R}$, then addition is continuous.

Second, we show that inversion is continuous. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Let $\delta = \varepsilon$. If $|x - x'| < \delta = \varepsilon$ then $|(-x) = (-x')| = |x' - x| = |x - x'| < \varepsilon$. Therefore inversion is continuous at x and since x is an arbitrary element of $\mathbb{R}$, then inversion is continuous. Therefore, $\mathbb{R}$ under addition form a topological group.

Example. (Exercise 22S.2(c)) The multiplicative group of positive real numbers $\mathbb{R}^+$ (under the subspace topology of the usual topology) is a topological group. Let $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$ and let $\varepsilon > 0$. Let $\delta_1 = \varepsilon/(2y)$ and $\delta_2 = \varepsilon/(2(x + \delta_1)) = y\varepsilon/(2x + y\varepsilon)$. Consider the open set $\mathcal{O} = \{x' \mid |x - x'| < \delta_1\} \times \{y' \mid |y - y'| < \delta_2\}$ in $\mathbb{R}^+ \times \mathbb{R}^+$. For all $(x', y') \in \mathcal{O}$ we have

$$\begin{aligned}
|xy - x'y'| &= |xy - x'y + x'y - x'y'| \\
&= |y(x - x') + x'(y - y')| \leq |y(x - x')| + |x'(y - y')| \\
&\qquad \text{by the Triangle Inequality for absolute value} \\
&= y|x - x'| + x'|y - y'| < y\delta_1 + x'\delta_2 \\
&< y\delta_(x + \delta_1)\delta_2 \text{ since } |x - x'| < \delta_1 \text{ implies } x' < x + \delta_1 \\
&< y\frac{\varepsilon}{2y} + (x + \delta_1)\frac{\varepsilon}{2(x + \delta_1)} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

So multiplication is continuous at (x, y) and since (x, y) is an arbitrary point of $\mathbb{R}^+ \times \mathbb{R}^+$, then multiplication is continuous.

Second, we show that inversion is continuous. Let $x \in \mathbb{R}^+$ and $\varepsilon > 0$. Without loss of generality, we take $0 < \varepsilon < 1/x$. Consider the open set $\mathcal{O} = \{x' \mid x/(1+x\varepsilon) < x' < x/(1 - x\varepsilon)\}$ in $\mathbb{R}^+$. For all $x' \in \mathcal{O}$ we have $\dfrac{1 - x\varepsilon}{x} < \dfrac{1}{x'} < \dfrac{1 + x\varepsilon}{x}$ or $\dfrac{1}{x} - \varepsilon < \dfrac{1}{x'} < \dfrac{1}{x} + \varepsilon$, so that $-\varepsilon < \dfrac{1}{x'} - \dfrac{1}{x} < \varepsilon$ and $\left|\dfrac{1}{x} - \dfrac{1}{x'}\right| < \varepsilon$. So inversion is continuous at x and since x is an arbitrary point of $\mathbb{R}^+$, then inversion is continuous. Therefore the positive reals $\mathbb{R}^+$ under multiplication form a topological group.

Example. (Exercise 22S.2(d)) The multiplicative group of complex numbers of modulus 1, $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, is is a topological group. Recall that any $z \in \mathbb{C}$ with $|z| = 1$ we have $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Also, if $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$ then $z_1 z_2 = e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. We take as a basis for a topology on S^1 the collection of all sets of the form $\mathcal{B} = \{z = e^{i\theta} \mid \theta \in (z, b)$ for some $z, b \in \mathbb{R}$ with $a < b\}$ (the elements of $\mathcal{B}$ are "open arcs" of the unit circle). Notice that $\mathcal{B}$ is in fact a basis for a topology by the definition of basis. The topology is induced by the metric d on S^1 defines as $d(z_1, z_2) = d(e^{i\theta_1}, e^{i\theta_2}) = |\theta_1 - \theta_2| (\text{mod } 2\pi)$ where

$$|z| (\text{mod } 2\pi) = \begin{cases} 2n\pi - x \text{ if } 2n\pi \le x \le (2n+1)\pi \text{ for some } n \in \mathbb{Z} \\ 2n\pi - x \text{ if } (2n-1)\pi < x < 2n\pi \text{ for some } n \in \mathbb{Z}. \end{cases}$$

To show multiplication is continuous let $z_1, z_2 \in S^1 \times S^1$ and let $\varepsilon > 0$. Consider the open set $\mathcal{O} = \{z_1' \mid |z_1 - z_1'| < \varepsilon/2\} \times \{z_2 \mid |z_2 - z_2'| < \varepsilon/2\}$. If $(z_1', z_2') \in \mathcal{O}$ then

$$\begin{aligned} d(z_1 z_2, z_1' z_2') &= d(e^{i\theta_1} e^{i\theta_2}, e^{i\theta_1'} e^{i\theta_2'}) \\ &= d(e^{i(\theta_1+\theta_2)}, e^{i(\theta_1'+\theta_2')}) \\ &= |(\theta_1 + \theta_2) - (\theta_1' + \theta_2')| (\text{mod } 2\pi) \\ &= |(\theta_1 - \theta_1') + (\theta_2 - \theta_2')| (\text{mod } 2\pi) \\ &\le |\theta_1 - \theta_1'| (\text{mod } 2\pi) + |\theta_2 - \theta_2'| (\text{mod } 2\pi) \\ &\qquad \text{by the Triangle Inequality for } |\cdot| (\text{mod } 2\pi) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

So multiplication is continuous at (z_1, z_2) and since (z_1, z_2) is an arbitrary point of $S^1 \times S^1$, then multiplication is continuous.

Second, we show that inversion is continuous. Let $z = e^{i\theta} \in S^1$ and $\varepsilon > 0$. Let

$\delta = \varepsilon$. If $d(z, z') < \delta = \varepsilon$ then

$$d(z^{-1}, (z')^{-1}) = d(e^{-i\theta}, e^{-i\theta'}) = |(-\theta)-(-\theta')| (\text{mod } 2\pi) = |\theta-\theta'| (\text{mod } 2\pi) = d(z, z') < \varepsilon.$$

Therefore inversion is continuous at z and since z is an arbitrary element of S^1, then inversion is continuous. Therefore, S^1 is a topological group.

Note. The *general linear group*, denoted $GL(m, \mathbb{R})$, consists of the multiplicative group of all invertible $n \times n$ matrices with real entries. It is given the subspace topology by considering it as a subset of R^{n^2}. Multiplication and inversion are continuous and so $GL(n, \mathbb{R})$ is a topological group (this is Exercise 22S.2(e)). this idea is generalized in real analysis to give the set of all invertible linear operators on a Banach space E, denoted $GL(E)$, is a topological group. See the Section 22.1 notes from Royden and Fitzpatick's book, mentioned above.

Note. A subgroup of $GL(n, \mathbb{R})$ is the *special linear group*, denoted $SL(n, \mathbb{R})$, consisting of all $n \times n$ matrices with determinant 1. Recall that $\det(AB) = \det(A)\det(B)$.

Note. Another subgroup of $GL(n, \mathbb{R})$ is the *orthogonal group*, denoted $O(n, \mathbb{R})$, consisting of all distance preserving $n \times n$ matrices. Such matrices, called *orthogonal matrices*, have as their inverse their transpose: $A^{-1} = A^T$. The orthogonal group is a topological group and as a topological space it is compact (so $O(n, \mathbb{R})$ is a "compact group").

Note. A subgroup of $O(n,\mathbb{R})$ is the *special orthogonal group*, denoted $SO(n,\mathbb{R})$. this group consists of orthogonal $n \times n$ matrices with determinant 1. Notice that $SO(n,\mathbb{R})$ is also a subgroup of $SL(n,\mathbb{R})$. These are also called the *rotation groups*. When $n = 2$, we get

$$SO(2,\mathbb{R}) = SO(2) = \left\{ \left[\begin{array}{cc} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{array} \right] \middle| n \in \mathbb{Z} \right\}.$$

This group is isomorphic to S^1 described above.

Note. Exercise 22S.3 states: "Let H be a subspace of G. Show that if H is also a subgroup of G, then both H and $\overline{H}$ are topological groups." So if we take a subgroup H of a topological group G and give it the subspace topology, then H itself is a topological group. Therefore the subgroups of $GL(n,\mathbb{R})$ given above, namely the special linear group $SL(n,\mathbb{R})$, the orthogonal group $O(n,\mathbb{R})$, and the special orthogonal group $SO(n,\mathbb{R})$, are each topological groups.

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