Chapter 3. Connectedness and Compactness

Section 23. Connected Spaces

Note. As usual, the approach to connectedness in the topological space setting is similar to the approach in senior level analysis. See the last page of: http://faculty.etsu.edu/gardnerr/4217/notes/3-1.pdf. Notice that in Theorem 3-14 it is shown that a set of real numbers is connected if and only it it is an interval or a singleton. This result will be proved in the class in the next section.

Definition. Let X be a topological space. A *separation* of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. Space X is *connected* if there is no separation of X.

Note. An alternative definition is that X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

Note. Since connectivity is defined in term of open sets only, if X is connected and Y is homeomorphic to X then Y is connected. That is, connectivity is a topological property.

Lemma 23.1. Let Y be a subspace of X. Two disjoint nonempty sets A and B whose union is Y form a separation of Y if and only if A contains no limit points of B and B contains not limit points of A.

Example 2. Let $Y = [-1, 0) \cup (0, 1] \subset \mathbb{R}$. Then $A = [-1, 0)$ and $B = (0, 1]$ form a separation of Y and so Y is not connected.

Example 4. The rational $\mathbb{Q}$ are not connected. For any $a \in \mathbb{R} \setminus \mathbb{Q}$, a separation of $\mathbb{Q}$ is given by $A = \mathbb{Q} \cap (-\infty, a)$ and $B = \mathbb{Q} \cap (a, \infty)$.

Note. We now give several results on how to build new connected sets from known connected sets. Since showing a set is connected is equivalent to showing that there is *not* a separation, the proofs are by contradiction.

Lemma 23.2. If sets C and D form a separation of X and if Y is a connected subspace of X, then Y lies entirely in either C or in D.

Theorem 23.3. The union of a collection of connected subspaces of X that have a point in common is connected.

Theorem 23.4. Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.

Note. The whole point of calling a function "continuous" is not because of ε's and δ's or because of inverse images of open sets, but because continuous functions take connected sets to connected sets. This property is proved in the next theorem.

Theorem 23.5. The image of a connected space under a continuous map is connected.

Theorem 23.6. A finite Cartesian product of connected spaces is connected.

Note. Theorem 23.6 holds for *finite* Cartesian products. The following example shows that it does not hold for an (countable) infinite product of connected spaces.

Example 6. Consider the Cartesian product $\mathbb{R}^\omega$ in the box topology. Then $\mathbb{R}^\omega$ is the union of set A consisting of all bounded sequences of real numbers and set B of all unbounded sequences. These sets are disjoint. Let $\mathbf{A} = (a_1, a_2, \ldots) \in \mathbb{R}^\omega$. Consider basis element (and hence open set) for the box topology $U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \cdots$. If $\mathbf{a} \in A$ is bounded then U is an open set containing $\mathbf{a}$ and consisting only of bounded sequences (and so $U \subset A$). So A is open. If $\mathbf{a} \in B$ is unbounded then U is an open set containing $\mathbf{a}$ and consisting only of unbounded sequences (and so $U \subset B$). So B is open. Therefore A and B form a separation of $\mathbb{R}^\omega$ and so $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots$ under the box topology is not connected even though (as will be proved in the next section) $\mathbb{R}$ is connected.

Example 7. Now consider $\mathbb{R}^\omega$ in the product topology. Let $\mathbb{R}^n$ denote the subspace of $\mathbb{R}^\omega$ consisting of all sequences $\mathbf{x} = (x_1, x_2, \ldots)$ such that $x_i = 0$ for $i > n$. The space $\tilde{\mathbb{R}}^n$ is "'clearly" homeomorphic to $\mathbb{R}^n$ and so (assuming $\mathbb{R}$ is connected) is connected by Theorem 23.6. The space $\mathbb{R}^\infty = \{\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\omega \mid x_i \neq 0$ for only finitely many $i \in \mathbb{N}\}$ is the union of the spaces $\tilde{\mathbb{R}}^n$, $\mathbb{R}^\infty = \cup_{n\in\mathbb{N}}\tilde{\mathbb{R}}^n$, and since $\mathbf{0} \in \tilde{\mathbb{R}}^n$ for all $n \in \mathbb{N}$, then $\mathbb{R}^\infty$ (under the product topology) is connected by Theorem 23.3.

We now show $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$. Let $\mathbf{a} = (a_1, a_2, \ldots) \in \mathbb{R}^\omega$. Let $U = \prod_{i\in\mathbb{N}} U_i$ be a basis element for the product topology that contains $\mathbf{a}$ (so $U_i = \mathbb{R}$ for all but finitely many i). So there is $N \in \mathbb{N}$ such that $U_i = \mathbb{R}$ for $i > N$. Then the point $\mathbf{x} = (a_1, a_2, \ldots, a_M, 0, 0, \ldots) \in \mathbb{R}^\infty$ belongs to U since $a_i \in U_i$ for $1 \leq i \leq N$ and $0 \in U_i$ for $i > N$. Since U is an arbitrary basis element for the product topology which contains $\mathbf{a}$ and we have shown that U contains an element of $\mathbb{R}^\infty$ then $\mathbf{a} \in \overline{\mathbb{R}^\infty}$. Since $\mathbf{a}$ is an arbitrary element of $\mathbb{R}^\omega$ then $\mathbb{R}^\omega \subset \overline{\mathbb{R}}$. Since $\mathbb{R}^\infty \subset \mathbb{R}^\omega$ it follows that $\mathbb{R}^\omega = \overline{\mathbb{R}^\infty}$. Since $\mathbb{R}^\infty$ under the product topology is connected as shown above, then by Theorem 23.4 (since $\mathbb{R}^\infty = \overline{\mathbb{R}^\infty}$), $\mathbb{R}^\omega$ (under the product topology) is connected.

Note. The argument given in Example 7 extends from a countable product of copies of $\mathbb{R}$ (under the product topology) to an arbitrary product of connected spaces under the product topology, as is shown in Exercise 23.10.

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