Section 24. Connected Subspaces of the Real Line

Note. In this section we prove that intervals in $\mathbb{R}$ (both bounded and unbounded) are connected sets. In a senior level analysis class, a bit more can be said: A set of real numbers is connected if and only if it is an interval or a singleton. See Theorem 3-14 of `http://faculty.etsu.edu/gardnerr/4217/notes/3-1.pdf`. Single point sets ("singletons") are trivially connected; an interval, by definition, has distinct endpoints if it is bounded (and an interval may have none, one, or two endpoints). We in fact study a more general structure than $\mathbb{R}$ called a linear continuum. A linear continuum lacks the algebraic (field) properties of $\mathbb{R}$.

Definition. A simply ordered set L having more than one element is a *linear continuum* if the following hold:

(1) L has the least upper bound property (i.e., every set with an upper bound has a least upper bound).

(2) If $x < y$, then there exists x such that $x < z < y$.

Theorem 24.1. If L is a linear continuum in the order topology, then L is connected and so are intervals and rays in L.

Note. Applying Theorem 24.1 to the linear continuum $\mathbb{R}$, we get the following.

Corollary 24.2. The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.

Note. The following is a familiar result from Calculus 1, but in the setting of connected topological spaces and ordered topological spaces.

Theorem 24.3. Intermediate Value Theorem.
Let $f : X \to Y$ be a continuum map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point $x \in X$ such that $f(c) = r$.

Example 1. Recall that the *ordered square*, I_o^2, is $I \times I = [0,1] \times [0,1]$ with the dictionary order topology (see Section 16, Example 3). This forms a linear continuum as well. It satisfies the least upper bound property for the following reasons.

Let $A \subset I \times I$. Let $\pi_1 : I \times I \to I$ be the projection into the first coordinate (see Section 15). Let $b = \sup \pi_1(A)$ ($\pi_1(A) \subset I$ is a subset of $\mathbb{R}$ and so satisfies the least upper bound property; notice that 1 is an under bound of $\pi_1(A)$). If $b \in \pi_1(A)$ then A intersects $\{b\} \times I$ (see Figure 24.3(a)). Since $\{b\} \times I$ is homeomorphic to I, then $A \cap (\{b\} \times I) \subset \{b\} \times I$ has upper bound $(b, 1)$, and so has the least upper bound of the form (b, c). (b, c) is also the least upper bound for A.

If $b \notin \pi_1(A)$ then $(b, 0)$ is the least upper bound of A because no element of the form (b', c) with $b' < b$ can be an upper bound for A, for then b' would be an upper bound for $\pi_1(A)$, contradicting the fact that $b > b'$ is the least upper bound of $\pi_1(A)$. So the ordered square has the least upper bound property.

The property that $x < y$ implies the existence of z with $x < z < y$ is "trivial" and left as Exercise 24.A.

Note. The concept of "path connected," defined next, might first seem to be equivalent to the definition of "connected" introduced earlier. However, we will see by example that this is not the case.

Definition. Given points x and y of the space X, a *path* in X from x to y is a continuous map $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$. A space X is *path connected* if every pair of points of X can be joined by a path in X.

Lemma 24.A. If space X is path connected then it is connected.

Note. We now give some examples of path connected spaces, but we also give examples of connected spaces which are not path connected, showing that the converse of Lemma 24.A does not hold.

Example 3. Define the *unit ball* B^n in $\mathbb{R}^n$ as $B^n = \{\mathbf{x} \mid \|\mathbf{x}\| \le 1\}$ where $\|\mathbf{x}\| = \|(x_1, x_2, \ldots, x_n)\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Then for $\mathbf{x}, \mathbf{y} \in B^n$ we have that $f : [0, 1] \to \mathbf{R}^n$ defined as $f(t) = (1 - t)\mathbf{x} + t\mathbf{y}$ satisfies $f([0, 1]) \subset B^n$ and so $f([0, 1])$ is a path from $\mathbf{x}$ to $\mathbf{y}$ and B^n is path connected. Similarly, for all $\mathbf{x} \in \mathbb{R}^n$ the sets $B(\mathbf{x}, \varepsilon)$ and $\overline{B}(\mathbf{x}, \varepsilon)$ are path connected for all $\varepsilon > 0$.

Example 4. Define the *punctuated Euclidean space*, $\mathbb{R}^n \setminus \{\mathbf{0}\}$. If $n > 1$, this space is path connected.

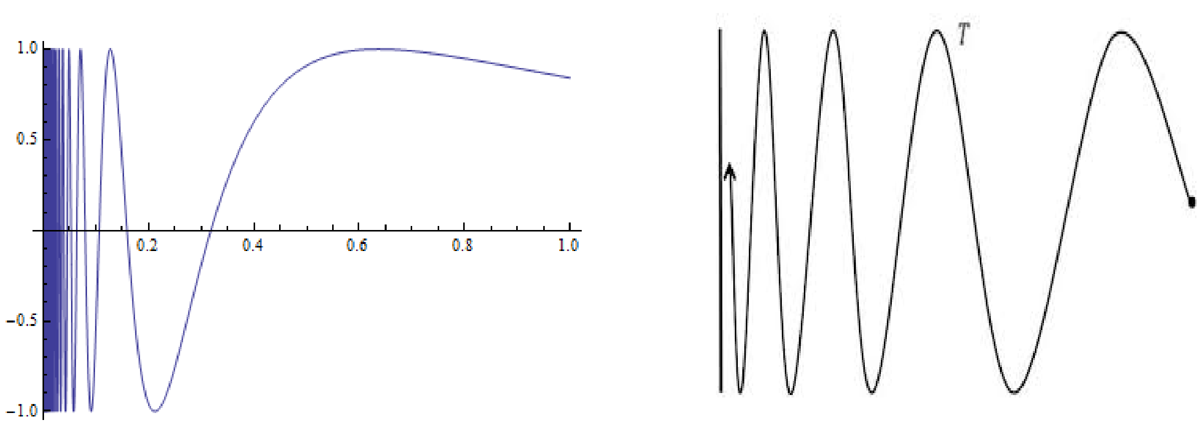
Example 6. We claim that the ordered square, I_o^2, is connected but not path connected. Connected is easy: By Example 1, I_o^2 is not a linear continuum and so by Theorem 24.1 is connected.

The fact that I_o^2 is not path connected is more complicated. Let $\mathbf{p} = (0,0)$ and $\mathbf{q} = (1,1)$. ASSUME there is a path $f : [a,b] \to I_o^2$ joining $\mathbf{p}$ and $\mathbf{q}$. Since every $\mathbf{x} \in I_o^2$ satisfies $\mathbf{p} \le \mathbf{x} \le \mathbf{q}$, then by The Intermediate Value Theorem (Theorem 24.3), the image set $f([a,b])$ must contain every point $\mathbf{x} = (x,y) \in I_o^2$. Therefore for each $\mathbf{x} \in I = [0,1]$, the set $U_x = f^{-1}(\{x\} \times (0,1))$ is a nonempty subset of $[a,b]$. By the definition of continuity of f, U_x is open. For each $x \in I = [0,1]$, choose a rational number $q_x \in U_x$. Since the sets U_x are disjoint, the map $x \mapsto q_x$ is a one-to-one (injective) mapping of I into $\mathbb{Q}$. But $|I| > |\mathbb{Q}|$, so this is a CONTRADICTION. So the assumption that I_o^2 is path connected is false.

Example 7. Let $S = \{(x, \sin(1/x)) \mid 0 < x \le 1\} \subset \mathbb{R}^2$. The mapping $x \mapsto (x, \sin(a/x))$ is continuous on $(0,1]$ by Theorem 18.4 (Maps Into Products). So by Theorem 23.5, set S is connected. So the closure $\overline{S} \subset \mathbb{R}^2$ is connected by Theorem 23.4. The set $\overline{S}$ is called the *topologist's sine curve.* In fact, $\overline{S} = S \cup (\{0\} \times [-1,1])$. ASSUME there is a path $f : [a,c] \to \overline{S}$ beginning at $(0,0)$ and ending at a point of S. The set

$$\{t \in [a,c] \mid f(t) \in \{0\} \times [-1,1]\} = f^{-1}(\{0\} \times [-1,1])$$

is closed by Theorem 18.1 (the (1)$\Rightarrow$(3) part) and so has a largest element b (WHY?). Then $f : [b, c] \to \overline{S}$ is a path that maps b into $\{0\} \times [-1, 1]$ and maps $(b, c]$ to S. WLOG $[b, c]$ can be replaced by $[0, 1]$ (by scaling and shifting t in $f(t)$). Let $f(t) = (x(t), y(t))$. Then $x(0) = 0$, $x(t) > 0$ for $t > 0$, and $y(t) = \sin(1/x(t))$ for $t > 0$. Construct a sequence $\{t_n\}$ as follows: For $n \in \mathbb{N}$, choose u with $0 < u_n < x(1/n)$ such that $\sin(1/u_n) = (-1)^n$. Since $\sin(1/x)$ is continuous on $(0, x(1/n))$, by the Intermediate Value Theorem (Theorem 24.3) there is some t_n with $0 < t_n < 1/n$ such that $x(t_n) = u_n$. But then $\{t_n\} \to 0$ (converges) and $f(t_n) = (x(t_n), \sin(1/x(t_n))) = (x(t_n), \sin(t/u_n) = (x(t_n), (-1)^n)$; so $\{f(t_n)\}$ does not converge. But since f is continuous, this behavior of $\{t_n\}$ and $\{f(t_n)\}$ CONTRADICTS The Sequence Lemma (Lemma 21.2).



A to-scale graph of the topologist's sine curve and a schematic. From `http://hyperspacewiki.org/images/1/15/Topologistsinecurve.png` and `http://faculty.mccneb.edu/akriesel/AmandaKrieselAPP3.pdf`, respectively.