Section 25. Components and Local Connectedness

Note. In senior level analysis, it is shown that an open set of real numbers consist of a countable number of maximal connected components which are themselves open intervals. See Theorem 3-5 of `http://faculty.etsu.edu/gardnerr/4217/notes/Supplement-Open-Sets.pdf`. In this section, we consider more generally connected components of topological spaces.

Definition. Given topological space X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called *components* (or "connected components") of X.

Note. We need to confirm that $\sim$ is actually an equivalence relation. Symmetry and reflexivity are clear. Transitivity follows from the fact that if A is connected and contains x and y, and B is connected and contains y and z, then $A \cup B$ contains x and z and $A \cup B$ is connected by Theorem 23.3. Therefore, $x \sim y$ and $y \sim z$ implies $x \sim z$.

Theorem 25.1. The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

Note. In addition to connected components we can also consider path connected components.

Definition. Define another equivalence relation on X as $x \sim y$ if and only if there is a path in X from x to y. The equivalence classes are called *path connected components* of X.

Note. It is a bit lengthier argument to show that this relation is an equivalence relation than it was to show the previous relation is an equivalence relation. First we note that if there is a path $f : [a, b] \to X$ from x to y then $g : [c, d] \to X$ defined as $g(t) - (t(c - d) + ad - bc)/)a - b)$ is a path from x to y with domain $[c, d]$. Now $\sim$ is reflexive since a constant path $f(t) = x$ is a path from x to x. For symmetry, if $x \sim y$ because $f : [0, 1] \to X$ is a path from x to y, then $g(t) = f(1-t)$ is a path from y to x and so $y \sim x$. For transitivity, if $x \sim y$ and $y \sim z$ then there is a path $f : [0, 1] \to X$ from x to y and a path $g : [1, 2] \to X$ from y to x (by the comment above, WLOG we can assume the given domains of f and g). So

$$h(t) = \begin{cases} f(t) & \text{for } t \in [0, 1] \\ g(t) & \text{for } t \in (1, 2] \end{cases}$$

is a path from x to z with domain $[0, 2]$ (we have used the Pasting Lemma, Theorem 18.3, for the continuity of h); therefore $x \sim z$ and $\sim$ is transitive. So $\sim$ is an equivalence relation.

Note. The proof of the following is similar to the proof of Theorem 25.1 and is left as Exercise 25.A.

Theorem 25.2. The path component of X are path-connected disjoint subspaces of X whose union is X, such that each nonempty path-connected subspace of X intersects only one of the path components.

Note. We now turn our attention to the topic of openness/closedness of components.

Lemma 25.A. Each connected component of a space X is closed. If X has only finitely many connected components, then each component of X is also open.

Note. In general, the connected components of a space may not be open, as shown now by example.

Example 1. Each connected component of $\mathbb{Q}$ is a singleton. The complement of a singleton is open, $\mathbb{Q} \setminus \{q\} = (\mathbb{Q} \cap (-\infty, q)) \cup (\mathbb{Q} \cap (q, \infty))$, and so each component is closed and not open (in the subspace topology where $\mathbb{Q} \subset \mathbb{R}$).

Example 2. The topologist's since curve, $\overline{S}$, of Section 24 is connected, as shown in Example 7 of that section. So $\overline{S}$ has one connected component. It was also shown that $\overline{S}$ is not path connected. However, $S = \{(x, \sin(1/x)) \mid x \in (0, 1]\}$ is the image of $(0, 1]$ under a continuous function and so is path connected. Also, $V = \{0\} \times [-1, 1]$ is similarly path connected. So $\overline{S}$ has two path-connected components, S and V. Notice that S is open (in the subspace topology) but not closed (it does not include its limit points in V). V is closed but not open. Munkres claims that if we omit all points from $\overline{S}$ by deleting all points of V having rational second coordinates, then we get a set with one connected component by uncountably many path components.

Definition. A space X is *locally connected at x* if for every neighborhood U of x, there is a connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is *locally connected* itself. Similarly, a space X is *locally path connected at x* if for every neighborhood U of x, there is a path-connected neighborhood V of x contained at each of its points, it is *locally path connected* itself.

Example 3. Each interval in $\mathbb{R}$ is both connected and locally connected. The subspace $[-1, 0) \cup (0, 1] \subset \mathbb{R}$ is not connected but it is locally connected. The topologist's sine curve, $\overline{S} \subset \mathbb{R}^2$, is connected but not locally connected because a neighborhood (in the subspace topology) U of point $x \in V = \{0\} \times [-1, 1] \setminus \{(0, 0)\}$ only contains neighborhoods V of x which are not connected because they contain "pieces" of set S.

Note. The following result classifies locally connected spaces.

Theorem 25.3. A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

Note. The proof of the following is similar to the proof of Theorem 25.3 and is left as Exercise 25.B.

Theorem 25.4. A space X is locally path connected if and only if for every open set U of X, each path component of U is open in X.

Note. The following result gives the relationship between path components and connected components.

Theorem 25.5. If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the component and the path components are the same.

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