

Section 26. Compact Sets

Note. You encounter compact sets of real numbers in senior level analysis shortly after studying open and closed sets. Recall that, in the real setting, a continuous function on a compact set attains a maximum and minimum (the Extreme Value Theorem) and a continuous function on a compact set is uniformly continuous. In addition, any sequence of elements of a compact set has a convergent subsequence (equivalent to the Bolzano-Weierstrass Theorem). For such results, see my notes for Analysis 1 (MATH 4217/5217) at: <http://faculty.etsu.edu/gardnerr/4217/notes.htm>. The (hopefully) familiar Heine-Borel Theorem states that a set of real numbers is compact if and only if it is closed and bounded (which we shall establish in Section 27). But beware that there are settings in which a set is closed and bounded but not compact! See, for example, my notes for Functional Analysis (MATH 5740): <http://faculty.etsu.edu/gardnerr/Func/notes/HWG-5-4.pdf>.

Note. The following historical notes on compact sets are based on

- *Analysis by Its History* in Springer-Verlag's Undergraduate Texts in Mathematics Readings in Mathematics, E. Hairer and G. Wanner, NY: Springer-Verlag (1996).
- The Wikipedia webpage on "Compact Spaces": https://en.wikipedia.org/wiki/Compact_space. (Accessed July 24, 2016.)

As Munkres states (see page 163), "From the beginnings of topology, it was clear that the closed interval $[a, b]$ of the real line had a certain property that was crucial

for proving such theorems as the maximum value theorem and the uniform continuity theorem. . . . It used to be thought that the crucial property of $[a, b]$ was the fact that every infinite subset of $[a, b]$ has a limit point, and this property was the one dignified with the name of compactness. Later, mathematicians realized that this formulation does not lie at the heart of the matter, but rather that a stronger formulation, in terms of open coverings of the space, is more central.”

In 1817, Bernard Bolzano (1781–1848) proved that any bounded sequence of real numbers has a subsequence which converges. His proof is based on dividing the bounded set in half and observing that one half or the other half must contain an infinite number of terms of the sequence, and then iterating this process. This is the same proof presented in Analysis 1 (see the notes for Section 2-3). In the 1860s, Karl Weierstrass (1815–1897) rediscovered this result, which became known as the Bolzano-Weierstrass Theorem [Wikipedia].

In 1870, Heinrich Eduard Heine (1821–1881) proved, with the use of the ε/δ definitions finally formalized by Weierstrass (completing the approach first introduced by Cauchy in the early 19th century) that a continuous function on a closed and bounded interval of real numbers is uniformly continuous. In his proof, Heine used a lemma which claimed that for any countable covering with open intervals of a closed and bounded interval, there exists a finite subcover. Near the end of the 19th century, Émile Borel (1871–1956) used a method similar to Heine’s to prove that a closed and bounded interval of real numbers has the property that every open cover has a finite subcover (i.e., in modern terminology, is “compact”) [Wikipedia, Hairer and Wanner page 283]. So the Heine-Borel Theorem states that a set of real numbers is compact if and only if it is closed and bounded. In 1906, Maurice

Fréchet (1878–1973) first used the term “compact” in a paper in 1904 and later used it in his 1906 dissertation. Fréchet used the definition mentioned by Munkres above [Wikipedia].

The Russian school of point-set topology, lead by Pavel Alexandrov (1896–1982) and Pavel Urysohn (1898–1924), spread the idea of compactness which we will use in our (very general) topological space setting [Wikipedia].



Heinrich Heine (1821–1881)



Émile Borel (1871–1956)

Images from the MacTutor History of Mathematics archive

Definition. A collection \mathcal{A} of subsets of a space X is a *covering* of X (or is said to *cover* X), if the union of the elements of \mathcal{A} is equal to X . Covering \mathcal{A} is an *open cover* of X if its elements are open subsets of X . If Y is a subspace of X , a collection \mathcal{A} of subsets of X *covers* Y if the union of its elements contains Y .

Definition. A space X is *compact* if every open cover \mathcal{A} of X contains a finite subcollection that also covers X (called a *finite subcover*).

Example 1. The real line \mathbb{R} is not compact since the open covering $\mathcal{A} = \{(n, n+1) \mid n \in \mathbb{Z}\}$ has no finite subcover since any finite subset of \mathcal{A} can contain only a finite number of elements of \mathbb{Z} .

Example A. The bounded open interval $(a, b) \subset \mathbb{R}$ is not compact since open covering $\mathcal{A} = \{(a + 1/n, b - 1/n) \mid n \in \mathbb{N}, 1/n < (b - a)/2\}$ has no finite subcover, since any finite subset of \mathcal{A} only covers $(a + 1/N, b - 1/N)$ for some $N \in \mathbb{N}$.

Example 2. Consider $X = \{0\} \cup \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$. Given any open covering \mathcal{A} of X , there is an element U of \mathcal{A} containing 0. The open set U contains all but finitely many of the points $1/n$. For each point of X not in U , choose an element of \mathcal{A} containing the point. Then U , along with these finite number of elements of \mathcal{A} , form a finite subcover of X . So X is compact.

Example 3. Any space X containing only finitely many points is necessarily compact.

Note. Notice that to show a space is *not* compact, you can just find an open covering with no finite subcover. However, to show a space *is* compact by definition, you must consider all possible open coverings.

Lemma 26.1. Let Y be a subspace of X . Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y .

Theorem 26.2. Every closed subspace of a compact space is compact.

Note. The following lemma and theorem give a property of compact Hausdorff spaces.

Lemma 26.4. If Y is a compact subspace of the Hausdorff space X and $x_0 \notin Y$, then there exists disjoint open sets U and V of X containing x_0 and Y , respectively.

Theorem 26.3. Every compact subspace of a Hausdorff space is closed.

Note. We can use Theorem 26.3 to show that some subspaces are not compact by showing that they are not closed.

Example 5. Since \mathbb{R} is Hausdorff, we know that any nonclosed subset of \mathbb{R} is not compact. For example, neither $(a, b]$, $[a, b)$, nor (a, b) are compact.

Example 6. The Hausdorff condition is necessary in Theorem 26.3. Consider the finite complement topology on \mathbb{R} (see Example 3 of Section 12) in which the open sets are all sets U for which $X \setminus U$ is either finite or is all of X . So the only closed sets are the finite sets and \mathbb{R} . But every subset of \mathbb{R} is compact (as you will show in Exercise 26.A); so there are compact subspaces which are not closed. This is possible because the finite complement topology on \mathbb{R} is not Hausdorff (as you will show in Exercise 26.A).

Note. The following result shows that a continuous function maps compact sets to compact sets. This complements Theorem 23.5 which states that a continuous function maps connected sets to connected sets. These two results give us the “Three Cs”: **C**ontinuity, **C**onected, and **C**ompact (notice that “Closed” is not one of the Cs).

Theorem 26.5. The image of a compact space under a continuous map is compact.

Note. The following result uses the theorems of this section to give a condition indicating that a continuous mapping is in fact a homeomorphism.

Theorem 26.6. Let $f : X \rightarrow Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Note. The following lemma allows us to address the compactness of a product of compact spaces.

Lemma 26.8. The Tube Lemma.

Consider the product space $X \times Y$ where Y is compact. If N is an open set of $X \times Y$ containing the slice $\{x_0\} \times Y$ of $X \times Y$, then N contains some “tube” $W \times Y$ about $\{x_0\} \times Y$, where W is a neighborhood of x_0 in X .

Theorem 26.7. The product of finitely many compact spaces is compact.

Note. In fact, a product of an arbitrary number of compact spaces is compact. This result is called the Tychonoff Theorem and is addressed in Section 37. The proof is involved and uses Zorn's Lemma and the Axiom of Choice, is given in Exercise 37.5).

Note. The following definition will lead to a condition equivalent to a space being compact.

Definition. A collection \mathcal{C} of subsets of X has the *finite intersection property* if for every finite subcollection $\{C_1, C_2, \dots, C_n\}$ of \mathcal{C} , the intersection $C_1 \cap C_2 \cap \dots \cap C_n$ is nonempty.

Theorem 26.9. Let X be a topological space. Then X is compact if and only if for every collection \mathcal{C} of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ for all elements of \mathcal{C} is nonempty.

Note. An easy corollary to Theorem 26.9 is the following.

Corollary 26.A. Let X be a compact topological space and let $C_1 \supset C_2 \supset \dots \supset C_n \supset C_{n+1} \supset \dots$ be a *nested sequence* of closed sets in X . If each C_n is nonempty, then $\bigcap_{n \in \mathbb{N}} C_n$ is nonempty.

Revised: 7/27/2016