Section 27. Compact Subspaces of the Real Line

Note. As mentioned in the previous section, the Heine-Borel Theorem states that a set of real numbers is compact if and only if it is closed and bounded. When reading Munkres, beware that when he uses the term "interval" he means a bounded interval. In these notes we follow the more traditional analysis terminology and we explicitly state "closed and bounded" interval when dealing with compact intervals of real numbers.

Theorem 27.1. Let X be a simply ordered set having the least upper bound property. In the order topology, each closed and bounded interval X is compact.

Corollary 27.2. Every closed and bounded interval $[a, b] \subset \mathbb{R}$ (where $\mathbb{R}$ has the standard topology) is compact.

Note. We can quickly get half of the Heine-Borel Theorem with the following.

Corollary 27.A. Every closed and bounded set in $\mathbb{R}$ (where $\mathbb{R}$ has the standard topology) is compact.

Note. The following theorem classifies compact subspaces of $\mathbb{R}^n$ under the Euclidean metric (and under the square metric). Recall that the standard topology on $\mathbb{R}$ is the same as the metric topology under the Euclidean metric (see Example 1 of Section 14), so this theorem includes and extends the previous corollary. Therefore we (unlike Munkres) call this the Heine-Borel Theorem.

Theorem 27.3. The Heine-Borel Theorem.
A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the Euclidean metric d or the square metric ρ.

Note. A word of warning. The Heine-Borel Theorem does not state that "compact" and "closed and bounded" are equivalent in any metric space. For example the space "little ell two" of all square summable sequences of real numbers,

$$\ell^2 = \left\{ (x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ for } i \in \mathbb{N} \text{ and } \sum_{i=1}^{\infty} x_i^2 < \infty \right\}$$

is a metric space with metric d where $d(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{\infty} (x_i - y_i)^2 \right)^{1/2}$. The set $C = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots\} = \{(1,0,0,\ldots), (0,1,0,\ldots, (0,0,1,0,\ldots), \ldots\}$ is closed and bounded, but not compact. Consider, for example, the open covering $\{B_d(\mathbf{e}_i, \sqrt{2}/2) \mid i \in \mathbb{N}\}$ of disjoint balls centered at the elements of set C has no proper subcover and so no finite subcover. The space ℓ^2 is "infinite dimensional"; in fact, the Heine-Borel Theorem only holds in certain finite dimensional spaces. In particular, the closed unit ball in compact in a *normed linear space* if and only if the space is finite dimensional. See my Introduction to Functional Analysis (MATH 5740) notes at: `http://faculty.etsu.edu/gardnerr/Func/notes/2-8.pdf` (see Theorem 2.34).

Note. In Calculus 1 you see the Extreme Value Theorem which claims that a continuous function on $[a, b] \subset \mathbb{R}$ attains a maximum and a minimum (in fact, this is the result that justifies the approach you take in solving all of those max/min problems). In senior level analysis, you see that this extends from $[a, b]$ to compact sets of real numbers. We now generalize it to continuous functions on topological spaces (where the domain is compact and the codomain is an ordered set).

Theorem 27.4. Extreme Value Theorem.
Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exists points $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.

Note. Before we prove that a continuous function on a compact space is uniformly continuous (where the spaces are metric spaces), we need two definitions and a preliminary lemma.

Definition. Let (X, d) be a metric space. Let A be a nonempty subset of X. For each $x \in X$ define the *distance from point x to set A* as $d(x, A) = \inf\{d(x, a) \mid a \in A\}$.

Lemma 27.A. Let (X, d) be a metric space and A a fixed subset of X. Then $D : X \to \mathbb{R}$ defined as $D(x) = d(x, A)$ is continuous.

Lemma 27.5. The Lebesgue Number Lemma.

Let $\mathcal{A}$ be an open covering of metric space (X, d). If X is compact, there is $\delta > 0$ such that for each subset B of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing B. The number $\delta > 0$ is a Lebesgue number for covering $\mathcal{A}$.

Definition. A function f for the metric space (X, d_X) to the metric space (Y, d_Y) is *uniformly continuous* if given $\varepsilon > 0$ there is $\delta > 0$ such that for every pair $x_0, x_1 \in X$ we have

$$d_X(x_0, x_1) < \delta \implies d_Y(f(x_0), f(x_1)) < \varepsilon.$$

Theorem 27.6. Uniform Continuity Theorem.

Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous on X.

Note. We now present a topological property concerning cardinality of a space. This result will allow us to show that $\mathbb{R}$ is uncountable.

Definition. If X is a space, a point $x \in X$ is an *isolated point* of X if the one-point set $\{x\}$ is open in X.

Theorem 27.7. Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.

Note. The creation of $\overline{V}_n$ where $x_n = f(n) \notin \overline{V}_n$ in the proof of Theorem 27.7 is similar to Cantor's Diagonalization Argument in the proof that $(0,1) \subset \mathbb{R}$ is uncountable. See my notes for Analysis 1 (MATH 4217/5217), Section 1-3: `http://faculty.etsu.edu/gardnerr/4217/notes/1-3.pdf`.

Note. Since a closed interval $[a,b] \subset \mathbb{R}$ is a compact set in $\mathbb{R}$ by Corollary 27.2 and has no isolated points ("clearly"), then we can apply Theorem 27.7 to show that $[a,b]$ is not countable. We therefore have the following.

Corollary 27.8. Every closed and bounded interval $[a,b] \subset \mathbb{R}$ is uncountable.

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