Section 28. Limit Point Compactness

Note. In this brief section we introduce two properties equivalent to compactness for metrizable spaces. One of the properties is stronger than compactness in a more general setting. We also introduce other examples of a nonmetrizable space.

Definition. A space X is *limit point compact* if every infinite subset of X has a limit point.

Note. The term "limit point compact" is due to Munkres (see page 179, line 3). Munkres comments that the property is sometimes also called "Fréchet compactness" or the "Bolzano-Weierstrass property." Recall that the Bolzano-Weierstrass Theorem states that a bounded infinite set of real numbers (or elements of $\mathbb{R}^n$) must have a limit point.

Theorem 28.1. Compactness implies limit point compactness, but not conversely.

Example 1. Let $Y = \{y_1, y_2\}$ and let the topology on Y consist of $\varnothing$ and Y. Consider $X = \mathbb{N} \times Y$ with the product topology where $\mathbb{N}$ has the discrete topology. Then for $A \subset X$, $A \neq \varnothing$, A has an element of the form (n, y_i). Any open set containing (n, y_i) also contains the point (n, y_j) where $j = 3 - i$, and so every nonempty $A \subset X$ has a limit point and so X is limit point compact. However, X is not compact since the covering of X by the open sets $U_n = \{n\} \times Y$ for $n \in \mathbb{N}$ has no subcollection covering X (and so no finite subcollection covering X).

Definition. Let X be a topological space. If $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in X and if $n_1 < n_2 < \cdots < n_i < \cdots$ is an increasing sequence of natural numbers, then the sequence $\{y_i\}_{i=1}^{\infty}$ defined as $y_i = x_{n_i}$ is a *subsequence* of the sequence $\{x_n\}$. The space X is *sequentially compact* if every sequence of points of X has a convergent subsequence.

Note. We now show that each of three three types of compactness are the same in metrizable spaces. We follow Munkres' proof, but break it into pieces, including two preliminary lemmas.

Lemma 28.A. Let X be metrizable. If X is also sequentially compact then the conclusion of the Lebesgue Number Lemma (Lemma 27.5) holds for X.

Lemma 28.B. Let X be metrizable. If X is also sequentially compact, then for all $\varepsilon > 0$ there exists a finite covering of X by open ε-balls.

Theorem 28.2. Let X be a metrizable space. Then the following are equivalent:

(1) X is compact.

(2) X is limit point compact.

(3) X is sequentially compact.

Note. Munkres gives another "less trivial" example of a space which is limit point compact but not compact. We review some definitions from set theory before stating the example.

Definition. (From Section 3) A relation Con a set A is an *ordered relation* (or *simple order* or *linear order*) if:

(1) (Comparability) For all $x, y \in A$ for which $x \neq y$, either xCy or yCx.

(2) (Nonreflexivity) For no $x \in A$ does xCx hold.

(3) (Transitivity) If xCy and yCx then xCz.

Example. $A = \mathbb{R}$ has order relations greater than $>$ and less than $<$. We could also take A as $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$ under either $>$ or $<$.

Definition. (From Section 10) A set A with order relation $<$ is *well-ordered* if every nonempty subset A has a smallest element.

Example. $A = \mathbb{R}$ under the usual less than, $<$, is NOT well-ordered. $A = \mathbb{N}$ under the usual less than IS well-ordered.

Note. The Well-Ordering Principle (see page 65 of Section 10) states that every set A has an order relation for which A is well-ordered. In fact, this property is equivalent to the Axiom of Choice (which Munkres takes as given and so observes that the Well-Ordering Principle can be proved; that's why Munkres calls it the "Well-Ordering Theorem").

Definition. (From Section 10) Let X be a well-ordered set. Given $\alpha \in X$, let $X_\alpha = \{x \mid x \in X \text{ and } x < \alpha\}$. This is called the *section* of X by α.

Lemma 10.2. There exists a well-ordered set A having a largest element Ω, such that the section S_Ω of A by Ω is uncountable but every other section of A is uncountable.

Definition. For well-ordered set A with largest element Ω as described in Lemma 10.2, the section S_Ω is a *minimal uncountable well-ordered set*. The well-ordered set $S_\Omega \cup \{\Omega\}$ is denoted $\overline{S}_\Omega$. We put the order topology on both S_Ω and $\overline{S}_\Omega$.

Theorem 10.3. If A is a countable subset of S_Ω, then A has an upper bound in S_Ω.

Example 2a. We claim that the space S_Ω is limit point compact. Let A be an infinite subset of S_Ω. Choose a countable infinite subset B of A (which can be done since the "smallest infinity" is countable infinite). Since $B \subset S_\Omega$ is countable, by Theorem 10.3 set B has an upper bound b in S_Ω. Let a_0 be the smallest element of S_Ω (which exists because S_Ω is well-ordered). Then $B \subset [a_0, b]$. S_Ω has the least upper bound property by Exercise 10.1 (in fact, the exercise shows that every well-ordered set has the least upper bound property), so by Theorem 27.1 the interval $[a_0, b]$ is compact. By Theorem 28.1, $[a_0, b]$ is limit point compact and so set $B \subset [a_0, b]$ has a limit point $x \in [a_0, b]$. Then x is also a limit point of A (since $[a_0, b] \subset A$). Therefore, S_Ω is limit point compact.

Example 2b. We claim that S_Ω is not compact. Munkres justifies this with the single claim that S_Ω has no largest element.

Note. Since the two examples given above, $X = \mathbb{N} \times \{y_1, y_2\}$ and $X = S_\Omega$, are of spaces which are limit point compact but not compact then, by Theorem 28.2, we see that these spaces are not metrizable.

Example 3. We claim that $\overline{S}_\Omega = S_\Omega \cup \{\Omega\}$ is not metrizable. Since $S_\Omega = \{x \mid x \in A, x < \Omega\}$, then $\Omega \in \overline{S}_\Omega$ is a limit point of S_Ω. But any sequence of elements of S_Ω is bounded by an element of S_Ω, and so Ω is not the limit of any sequence of elements of S_Ω. By The Sequence Lemma (lemma 21.2), $\overline{S}_\Omega$ is not metrizible (the "converse" part).

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