Section 29. Local Compactness

Note. You may be familiar with a property that holds "locally" versus a property that holds "globally." The property of a function being Lipschitz on a set in contrast to the function be locally Lipschitz at points of the set is one example (see my supplemental Complex Analysis 1 notes on Lipschitz functions: `http://faculty.etsu.edu/gardnerr/5510/CSPACE.pdf`). Analogously, a topological space is locally compact if it satisfies a certain condition at each point of the space. We also introduce the idea of a one-point compactification and discuss it in the setting of the Riemann sphere.

Definition. Trivially, a compact set if locally compact.

Definition. A topological space X is *locally compact at point x* if there is some compact subspace X of X that contains a neighborhood of x. If X is locally compact at each of its points, set X is *locally compact.*

Example 1. $\mathbb{R}$ is locally comapct since $x \in \mathbb{R}$ lies in neighborhood $(x-1, x+1)$ which is in the compact space $[x-1, x+1]$. In Exercise 29.1, you will show that $\mathbb{Q}$ is not locally compact.

Example 2. Similar to the argument of $\mathbb{R}$, we have that $\mathbb{R}^n$ is locally compact. For $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{x}$ is in the basis element $(x_1-1, x_1+1) \times (x_2-1, x_2+1) \times \cdots \times (x_n-1, x_n+1)$

which in turn is contained in compact subspace $[x_1 - 1, x_1 + 1] \times [x_2 - 1, x_2 + 1] \times \cdots \times [x_n - 1, x_n + 1]$. However, $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots$ under the product topology is not locally compact. Recall that basis elements for the product topology are of the form $B = (a_1, b_1) \times (a_1, a_2) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \cdots$ (by Theorem 19.1). If C is a compact subspace of $\mathbb{R}^\omega$ that contains $x \in \mathbb{R}^\omega$ and there is a neighborhood of $\mathbf{x}$ in C, then the neighborhood contains a basis element of the form of B. But then $\overline{B} = [a_1, b_1] \times [a_1, a_2] \times \cdots \times [a_n, b_n] \times \mathbb{R} \times \mathbb{R} \times \cdots$ is a closed subspace of C and so would be compact by Theorem 26.2. But $\overline{B}$ is not compact (consider an open cover which requires all of the open sets to cover one of the $\mathbb{R}$ components of B). So C cannot be compact and $\mathbb{R}^\omega$ is not locally compact.

Example 3. Every simple ordered set X having the least upper bound property is locally compact since a basis element for X is of the form (a, b), $[a, b)$, or $(a, b]$ (see the definition of "order topology" in Section 14). The closure of any basis element is then a closed interval which, by Theorem 27.1, is compact. So X is locally compact (see the previous example for details relating basis elements to neighborhoods).

Note. On page 183, Munkres declares: "Two of the most well-behaved classes of spaces to deal with in mathematics are the metrizable spaces and the compact Hausdorff spaces." With an eye towards subspaces of these types of spaces, the following result gives conditions under which a space is homeomorphic to a subspace of a compact Hausdorff space.

Theorem 29.1. Let X be a topological space. Then X is a locally compact Hausdorff space if and only if there is a topological space Y satisfying the following conditions:

(1) X is a subspace of Y.

(2) The set $Y \setminus X$ consists of a single point.

(3) Y is a compact Hausdorff space.

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

Note. With the notation of Theorem 29.1, if X is compact then $Y \setminus X = \{\infty\} \in T_2$ and so $\{\infty\}$ is open and ∞ is (by definition) an isolated point. If X is not compact, then $\{\infty\} = Y \setminus X$ is not open in Y. So any open set in Y containing ∞ is a limit point of X and so $\overline{X} = Y$.

Definition. If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is a *compactification* of X. If $Y \setminus X$ is a single point, then Y is the *one-point compactification* of X ("the" because of the homeomorphism property of Theorem 29.1).

Note. By Theorem 29.1 and the note above, X has a one-point compactification Y is and only if X is a locally compact Hausdorff space that is not compact itself.

Example 4. The one-point compactification of $X = \mathbb{R}$ is homeomorphic to the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, as you will show in Exercise 29.6. As topological spaces, $\mathbb{R}^2$ and $\mathbb{C}$ are homeomorphic (with both having the metric topology). The one-point compactification of $\mathbb{R}^2$ and $\mathbb{C}$ is homeomorphic to the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. The topological space $\mathbb{C} \cup \{\infty\}$ is called the *Riemann sphere* or the *extended complex plane*. A metric d of the extended complex plane $\mathbb{C} \cup \{\infty\}$ is

$$d(z, z') = 2|z - z'|/\sqrt{(1 + |z|^2)(1 + |z'|^2)} \text{ and } d(z, \infty) = 2/\sqrt{1 + |z|^2}$$

for $z, z' \in \mathbb{C}$. Notice that the maximum distance between to points is 2. This is related to the fact that the diameter of the Riemann sphere is 2. This metric induces the topology on $\mathbb{C} \cup \{\infty\}$ described in Theorem 29.1 For details about the metric and some related projections, see my Complex Analysis 1 (MATH 5510) notes: `http://faculty.etsu.edu/gardnerr/5510/notes/I-6.pdf`.

Note. The following result classifies locally compact Hausdorff spaces. Unlike our original definition, this involves arbitrary neighborhoods of a given point.

Theorem 29.2. Let X be a Hausdorff space. Then X is locally compact if and only if given $x \in X$, and given a neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Note. The next corollary gives a way to recognize some locally compact subspaces of locally compact Hausdorff spaces.

Corollary 29.3. Let X by locally compact and Hausdorff. Let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.

Note. The final corollary allows us to embed locally compact Hausdorff spaces in compact Hausdorff spaces.

Corollary 29.4 A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact and Hausdorff.

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