Section 30. The Countability Axioms

Note. In Section 21, we encountered the concept of a topological space being first-countable at some point $x \in X$. In this section, we restate this definition and introduce a new "countability axiom." Both of the countability axioms involve countable (versus uncountable) bases of topologies.

Definition. A topological space X has a *countable basis at point* x if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of the points satisfies the *First Countability Axiom*, or is *first-countable*.

Note. As commented in Section 21, a metrizable space X is first-countable since $\mathcal{B} = \{B_d(x, 1/n) \mid n \in \mathbb{N}\}$ satisfies the definition for each $x \in X$ (with metric d).

Note. As observed in Section 21, if "metrizable" is replaced with "first-countable" in Lemma 21.2 (The Sequence Lemma) and Theorem 21.3, the results still hold (as seen in the proof of each). So these results can be restated as follows.

Theorem 30.1. Let X be a topological space.

(a) Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is first-countable.

(b) Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.

Definition. If a topological space X has a countable basis for its topology, then X satisfies the *Second Countability Axiom*, or is *second-countable.*

Note. Of course, if a space is second-countable then it is first-countable. So second-countable is more restrictive than first-countable. In fact, there are metric spaces which are not second countable (as we will see, $\mathbb{R}^\omega$ under the uniform topology is such an example; see Example 2). We will need second-countable for the proof of the Urysohn Metrization Theorem in Section 34.

Example 1. The real line $\mathbb{R}$ (under the standard topology) is first-countable; consider $\mathcal{B} = \{(a, b) \mid a < b, a, b \in \mathbb{Q}\}$. Similarly, $\mathbb{R}^n$ is first-countable; let $\mathcal{B}$ be all products of open intervals with rational endpoints. Even $\mathbb{R}^\omega$ (with the product topology) is second-countable; let $\mathcal{B}$ be all products $\prod_{n\in\mathbb{N}} U_n$ where finitely many U_n are open intervals with rational endpoints and the remaining $U_n = \mathbb{R}$.

Example 2. Consider $\mathbb{R}^\omega$ under the uniform topology (the topology induced by the uniform metric $\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\overline{d}(x_i, x_j) \mid i, j \in \mathbb{N}\}$ and $\overline{d}(x, y) = \min\{|x - y|, 1\}$ is the standard bounded metric on $\mathbb{R}$), which is a metric space. Since $\mathbb{R}^\omega$ is a metric space, it is (as observed above) first-countable. However, it is not second-countable. To establish this, we first show that if a space X has a countable basis $\mathcal{B}$ then any subspace A which has the discrete topology (under the subspace topology) must be countable. Under these conditions for X, $\mathcal{B}$, and A, for each $a \in A$ there is a basis element B_a that intersects A at point a alone (since set $\{a\}$ is open in the discrete topology). So if $a \neq b$ for $a, b \in A$ then corresponding B_a and B_b are different ($a \in B_1$ but $a \not\in B_b$, say). So the mapping $a \to B_a$ is one to one and $|A| \leq |B|$. So A must be countable. But subspace A of $\mathbb{R}^\omega$ consisting of all sequences of 0's and 1's is uncountable (map it to $[0, 1]$ using a binary representation of the elements of $[0, 1]$) and it has the discrete topology since the uniform metric gives $\overline{\rho}(a, b) = 1$ for any two distinct $a, b \in A$. So $\mathbb{R}^\omega$ cannot have a countable basis.

Theorem 30.2. A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable. A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.

Definition. A subset A of a space X is *dense* in X if $\overline{A} = X$.

Note. The set of rationals $\mathbb{Q}$ is dense in $\mathbb{R}$. The set of irrationals $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$. In fact, the set of algebraic numbers $\mathbb{A}$ is also dense in $\mathbb{R}$.

Theorem 30.3. Suppose X has a countable basis. Then:

(a) Every open covering of X contains a countable subcover.

(b) There exists a countable subset of X that is dense in X.

Note. A topological space for which every open cover has a countable subcover is often called a *Lindelöff space*. Since $\mathbb{R}$ is second-countable then $\mathbb{R}$ is a Lindelöf space, by Theorem 30.3(a) (see Theorem 3-9 of `http://faculty.etsu.edu/gardnerr/4217/notes/3-1.pdf`). A topological space with a countable dense subset is called *separable* (not to be confused with the concept of a "separation" from Section 23). This is an important property of "Hilbert spaces" for which separability allows the introduction of a countable basis. (see Theorem 5.4.8 of `http://faculty.etsu.edu/gardnerr/Func/notes/HWG-5-4.pdf`).

Note. In a metrizable space, the two conditions of Theorem 30.3 (Lindelöf and separable, respectively) are each equivalent to second-countable, as shown in Exercise 30.5. The following example shows the existence of a space which is first-countable, Lendelöf, and separable, but it is not second-countable (and so is not metrizable).

Example 3. Consider $\mathbb{R}_\ell$, $X = \mathbb{R}$ with the lower limit topology which has basis $\{[a, b) \mid a < b, a, b \in \mathbb{R}\}$. Given $x \in \mathbb{R}_\ell$, the set of all basis elements of the form $\{[x, x + 1/n) \mid n \in \mathbb{N}\}$ is a countable basis at x and so $\mathbb{R}_\ell$ is *first-countable.* Also, the rationals $\mathbb{Q}$ are dense in $\mathbb{R}_\ell$, so $\mathbb{R}_\ell$ is *separable.*

Let $\mathcal{B}$ be a basis for $\mathbb{R}_\ell$. For each $x \in \mathbb{R}_\ell$, there is some $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subset [x, x + 1)$. If $x \neq y$ then $B_x \neq B_y$ (since $x = \inf(B_x)$ and $y = \inf B_y$). So the mapping $x \to B_x$ of $\mathbb{R}_\ell$ onto $\mathcal{B}$ is one to one and hence $|\mathcal{B}| = |\mathbb{R}_\ell|$ and $\mathcal{B}$ is uncountable. That is, $\mathbb{R}_\ell$ is *not second-countable.*

Now we show that $\mathbb{R}_\ell$ is Lindelöf. We do so by showing that every open covering of $\mathbb{R}_\ell$ by basis elements has a countable subcover (if we start with an arbitrary open cover, for each x is one of the open sets there is a basis element containing x which is a subset of the open set; we can then convert the countable subcover of basis elements back into a countable subcover of the original covering). So let $\mathcal{A} = \{[a_\alpha, b_\alpha) \mid \alpha \in J\}$ be a covering of $\mathbb{R}$ by basis elements for the lower limit topology. Let $C = \cup_{\alpha \in J}(A_\alpha, b_\alpha)$ so that $C \subset \mathbb{R}$. We first show that $\mathbb{R} \setminus C$ is countable. Let $x \in \mathbb{R} \setminus C$. Then x is in no (a_α, b_α), so we must have $x = a_\beta$ for some $\beta \in J$ and so $x \in [a_\beta, b_\beta)$. Choose some $q_x \in \mathbb{Q}$ with $q_x \in (a_\beta, b_\beta)$. Since $(x, q_x) = (a_\beta, q_x) \subset (a_\beta, b_\beta) \subset C$, then for $x, y \in \mathbb{R} \setminus C$ with $x < y$ we have $q_x < q_y$ (otherwise we would have $x < y < q_y \leq q_x$ and $y \in (x, q_x) \subset C$, a contradiction). Therefore the map $x \mapsto q_x$ of $\mathbb{R} \setminus C$ into $\mathbb{Q}$ is one to one. So $|\mathbb{R} \setminus C| \leq |\mathbb{Q}|$ and $\mathbb{R} \setminus C$ is countable. We second show that $\mathcal{A}$ has a countable subcover. For each element of countable $\mathbb{R} \setminus C$, choose an element of $\mathcal{A}$ containing it. Combining all such elements of $\mathcal{A}$ yields a countable subcollection $\mathcal{A}'$ of $\mathcal{A}$ that covers $\mathbb{R} \setminus C$. Now C is a union of open intervals in $\mathbb{R}$ and so is an open set in $\mathbb{R}$; since $\mathbb{R}$ is

Lindelöf then there is a countable subcover of C, $(a_{\alpha_1}, b_{\alpha_1}), (a_{\alpha_2}, b_{\alpha_2}), \ldots$. Define $\mathcal{A}'' = \{[a_\alpha, b_\alpha) \mid \alpha = \alpha_1, \alpha_2, \ldots\}$. Then $\mathcal{A}'' \subset \mathcal{A}$ is a countable covering of C and so $\mathcal{A}' \cup \mathcal{A}''$ is a countable subcover of $\mathbb{R}_\ell$. That is, $\mathbb{R}_\ell$ is *Lindelöf.*

Example 4. In this example we show that a product of two Lindelöf spaces may not be Lindelöf. We saw in the previous example that $\mathbb{R}_\ell$ ($X = \mathbb{R}$ under the lower limit topology) is Lindelöf. Consider $\mathbb{R}_\ell \times \mathbb{R}_\ell = \mathbb{R}_\ell^2$ under the product topology (this topological space is called the *Sorgenfrey plane*). A basis for $\mathbb{R}_\ell^2$ is $\mathcal{B} = \{[a, b) \times [c, d) \mid a, b, c, d \in \mathbb{R}, a < b, c < d\}$. To show that $\mathbb{R}_\ell^2$ is not Lindelöf, we consider the subspace $L = \{(x, -x) \mid x \in \mathbb{R}_\ell\}$ (this is geometrically the line $y = -x$). Notice that $L \subset \mathbb{R}_\ell^2$ is closed and so $\mathbb{R}_\ell^2 \setminus L$ is open. Now we cover L with the basis elements $\mathcal{A}' = \{[a, a+1) \times [-a, -a+1) \mid a \in \mathbb{R}\}$. Notice that each of these basis elements intersects L in exactly one point: $[a, a+1) \times [-a, -a+1) \cap L = \{(a, -a)\}$. So there is no proper subset of $\mathcal{A}'$ which covers L. Then $\mathcal{A} = \mathcal{A}' \cup \{\mathbb{R}_\ell^2 \setminus L\}$ is an open covering of $\mathbb{R}_\ell^2$ with no countable subcover (in fact, no proper subcover) because L is uncountable ($|L| = |\mathbb{R}|$). so $\mathbb{R}_\ell^2$ is *not Lindelöf.*

Example 5. In this example we show that a subspace of a Lindelöf space may not be Lindelöf. Recall that I_0^2 is the ordered square $[0, 1] \times [0, 1]$ under the order topology induced by the dictionary order. In this topology (which is different from the subspace topology on $[0, 1] \times [0, 1]$ as a subspace of $\mathbb{R} \times \mathbb{R}$ with the dictionary order topology; see page 90), $I_0^2 = [(0,), (1, 1)]$, a closed interval. By Theorem 27.1, I_0^2 is compact (and therefore Lindelöf; finite subcovers are certainly countable).

However, consider the subspace $A = [0,1] \times (0,1)$. An open covering of A is given by $\{U_x\}_{x \in [0,1]}$ where $U_x = \{x\} \times (0,1)$ (notice that U_x is the open interval $((x,0),(x,1))$. Then the covering is uncountable and there is no proper subcover (and hence no countable subcover). Hence subspace A is not Lindelöf, where space I_0^2 is Lindelöf.

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