Section 31. The Separation Axioms

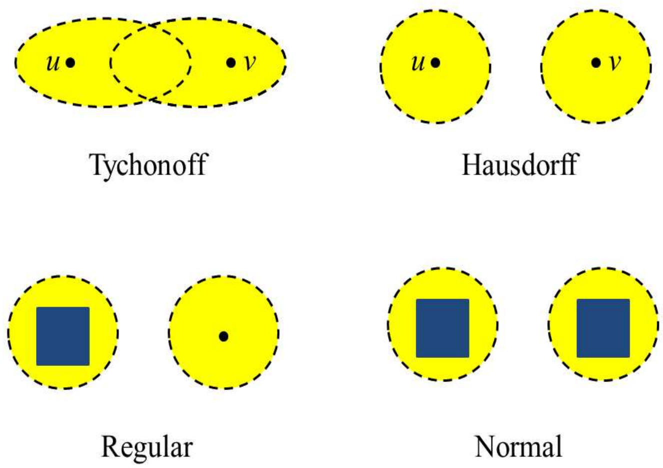
Note. Recall that a topological space X is Hausdorff if for any $x, y \in X$ with $x \neq y$, there are disjoint open sets U and V with $x \in U$ and $y \in V$. In this section, Munkres introduces two more separation axioms (we introduce a third).

Definition. A topological space X satisfies the *Tychonoff Separation Property* if for any $x, y \in X$ with $x \neq y$, there are open sets U and V with $x \in U$, $y \notin U$, and $y \in V$, $x \notin V$.

Note. A space is Tychonoff if and only if every singleton (one-point set) is a closed set, as you will show in Exercise 31.A. Notice that all Hausdorff spaces are Tychonoff (Hausdorff adds the "disjoint" condition on U and V). In the following definition, the first statement could be replaced with the Tychonoff Separation Property (see my notes based on Royden and Fitzpatrick's *Real Analysis*, 4th Edition, Section 11.2 "The Separation Properties": `http://faculty.etsu.edu/gardnerr/5210/notes/11-2.pdf`).

Definition. Suppose that one-point sets are closed in space X. Then X is *regular* if for each pair consisting of a point x and closed set B disjoint from $\{x\}$, there are disjoint open sets U and V with $x \in U$ and $B \subset V$. The space X is *normal* if for each pair A, B of disjoint closed sets of X, there are disjoint open sets U and V with $A \subset U$ and $B \subset V$.

Note. The four separation properties can be illustrated as follows:



We have the following schematics inclusions:

Normal Spaces $\subset$ Regular Spaces $\subset$ Hausdorff Spaces $\subset$ Tychonoff Spaces.

We show by example below that the first two inclusions are proper.

Note. We saw the "T_1 Axiom" in section 17 (namely, the property/axiom that finite point sets are closed). So the T_1 property is what we just labeled "Tychonoff." Sometimes this notation is used and a Hausdorff space is labeled "T_2," a regular space is "T_3" and a normal space is "T_4." An additional (weaker) separation axiom called "T_0" requires: For any $x, y \in X$, there is an open set U such that both $x \in U$ and $y \notin U$, or both $y \in U$ and $x \notin U$. To complicate things (!), T_0 spaces are sometimes called Kolmogorov spaces, T_1 spaces are Fréchet spaces, T_2 spaces are Hausdorff spaces, T_3 spaces are Vietoris spaces, and T_4 spaces are Tietze spaces. The inclusion above extends to these new categories to give (schematically): $T_4 \subset T_3 \subset T_2 \subset T_1 \subset T_0$. This information is from the reputable website *Wolfram MathWorld*: `http://mathworld.wolfram.com/SeparationAxioms.html` (accessed 8/17/2016). The "T" comes from the German "Trennungsaxiom" which means "separation axiom" (see Munkres' page 211).

Note. The following lemma gives a classification of regular and normal spaces.

Lemma 31.1. Let X be a topological space. Let one-point sets (singletons) in X be closed.

(a) X is regular if and only if given a point $x \in X$ and a neighborhood U of X, there is a neighborhood V of x such that $\overline{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Note. The following theorem considers subspaces and products of Hausdorff and regular spaces. The story with regular spaces is more complicated.

Theorem 31.2.

(a) A subspace of a Hausdorff space is Hausdorff. A product of Hausdorff spaces is Hausdorff.

(b) A subspace of a regular space is regular. A product of regular spaces is regular.

Note. There is no result for normal spaces analogous to the results of Theorem 31.2. Products of normal spaces are addressed in Examples 2 and 3 below and in Examples 1 and 2 of the next section.

Example 1. Recall that the K-topology on $\mathbb{R}$ (see page 82) has as its basis $\mathcal{B}^n = \{(a,b), (a,b) \setminus K \mid a, b \in \mathbb{R}, a < b\}$ where $K = \{1/n \mid n \in \mathbb{N}\}$. This space is denoted $\mathbb{R}_K$. Then $\mathbb{R}_K$ is Hausdorff since it includes all open (a,b).

Now set K is closed because it contains its limit points (of which there are none; see Corollary 17.7): any point in $\mathbb{R}\setminus(K\cup\{0\})$ is in a neighborhood of the form (a,b) not intersecting K, and $(-1,1)\setminus K$ is a neighborhood of 0 which does not intersect K. We consider closed set K and point $x = 0$ to show that $\mathbb{R}_K$ is not regular. ASSUME there exist disjoint open sets U and V containing 0 and K, respectively. Then there is a basis element N containing 0 lying in U. This basis element must be of the form $N = (a,b) \setminus K$, since each basis element of the form (a,b) containing 0 intersects K. Choose $n \in \mathbb{N}$ large enough that $1/n \in (a,b)$ (where $(a,b)\setminus K = N$). Since $1/n \in K \subset V$ and V is open, then there is a basis element $M \subset V$ containing $1/n$. Then M must be of the form (c,d), since $1/n \in M$. But then M contains some $z \in N \subset U$ (namely, any z such that $\max\{c, 1/(n+1)\} < z < 1/n$). So $z \in U \cap V$ and U and V are not disjoint, a CONTRADICTION. So the assumption of the existence of such U and V is false and $\mathbb{R}_K$ is not regular. So $\mathbb{R}_K$ is an example of a nonregular Hausdorff space showing that the set of regular spaces is a proper subset of the set of Hausdorff spaces.

Example 2. In this example, we show that $\mathbb{R}_\ell$ is normal. By Lemma 13.4, the topology of $\mathbb{R}_\ell$ is finer than the standard topology on $\mathbb{R}$ so one-point sets are closed in $\mathbb{R}_\ell$ Let A and B be disjoint closed sets in $\mathbb{R}_\ell$. For each $a \in A$, there is a basis element for the lower limit topology $[a, x_a)$ not intersecting B (since B is closed, it contains its limit points by Corollary 17.7 and s a is not a limit point of B since

$a \notin B$). Similarly, for each $b \in B$ there is a basis element $[b, x_b)$ not intersecting A. Then

$$U = \cup_{a \in A}[a, x_a) \text{ and } V = \cup_{b \in B}[b, x_b)$$

are disjoint open sets with $A \subset U$ and $B \subset V$. So $\mathbb{R}_\ell$ is normal.

Example 3. I this example we show that the Sorgenfrey plane $\mathbb{R}_\ell^2$ (see Example 4 of Section 30) is not normal. Since $\mathbb{R}_\ell^2 = \mathbb{R}_\ell \times \mathbb{R}_\ell$, this example (combined with the previous) shows that a product of normal spaces may not be normal. Also, since $\mathbb{R}_\ell$ is regular by the previous example (normal spaces are regular) then $\mathbb{R}_\ell^2 = \mathbb{R}_\ell \times \mathbb{R}_\ell$ is regular by Theorem 31.2(b). So the Sorgenfrey plane $\mathbb{R}_\ell^2$ is an example of a regular space which is not normal, showing that the set of normal spaces is a proper subset of the set of regular spaces.

ASSUME $\mathbb{R}_\ell^2$ is normal. Let L be the subspace of $\mathbb{R}_\ell^2$ consisting of all points of the form $(x, -x)$ (geometrically, the line $y = -x$). Now L is closed in $\mathbb{R}_\ell^2$ and has the discrete topology (as in example 4 of Section 30, every one-point set in L is open: $(a, -a) = L \cap ([a, a+1) \times [-a, -a+1))$). So every subset $A \subset L$ is closed in L (and open!) and by Theorem 17.3 is closed in $\mathbb{R}_\ell^2$. So $L \setminus A$ is also closed in $\mathbb{R}_\ell^2$. Since we assumed $\mathbb{R}_\ell^2$ is normal, there are disjoint open sets U_A and V_A such that $A \subset U_A$ and $L \setminus A \subset V_A$.

Let D be the set of points of $\mathbb{R}_\ell^2$ having rational coordinates. Then D is dense in $\mathbb{R}_\ell^2$ (any basis element containing a point in $\mathbb{R}_\ell^2 \setminus D$ contains points in D so all points in $\mathbb{R}_\ell^2 \setminus D$ are limit points of D and $\overline{D} = \mathbb{R}_\ell^2$). Define $\theta : \mathcal{P}(L) \to \mathcal{P}(D)$ ("$\mathcal{P}$"

for power set) as

$$\begin{aligned}\theta(A) &= D\cap U_A \text{ if } \varnothing \subsetneq A \subsetneq L\\ \theta(\varnothing) &= \varnothing\\ \theta(A) &= D.\end{aligned}$$

Let A and B be two different proper, nonempty subsets of L. Then $\theta(A) = D\cap U_A$ is neither empty (since D is dense in $\mathbb{R}^2_\ell$ and U_A is open; open U_A must intersect D and $\mathbb{R}^2_\ell \setminus D$) nor all of D (for if $D\cap U_A = D$ then $D\subset U_A$ and since $U_A\cap V_A = \varnothing$ we would have $D\cap V_A = \varnothing$, contradicting the fact that D is dense and V_A is open). Since sets $A\neq B$, then there is some $x\in\mathbb{R}^2_\ell$ in one set but not the other; say $x\in A$, $x\notin B$. Then $x\in L\setminus B$ and so $x\in A\subset U_A$ and $x\in L\setminus B\subset V_B$. That is, $x\in U_A\cap V_B$. But $U_A\cap V_B$ is a nonempty open set and so must contain points in dense set D. Such points are in U_A and not in U_B (since such points are in V_B and $U_B\cap V_B=\varnothing$). The existence of these points (though we have not CONSTRUCTED them) shows that there are points in $D\cap U_B$ which are not in $D\cap U_B$. That is,

$$\theta(A) = D\cap U_A \neq D\cap U_B = \theta(B).$$

So, $\theta : \mathcal{P}(L)\to\mathcal{P}(D)$ is one to one (injective).

We now construct a one to one map $\varphi : \mathcal{P}(D)\to L$. "Since D is countably infinite and L has the cardinality of $\mathbb{R}$" (Munkres, page 198), it suffices to define a one to one map ψ of $\mathcal{P}(\mathbb{N})$ into $\mathbb{R}$. For $S\in\mathcal{P}(\mathbb{N})$ (so $S\subset\mathbb{N}$) define $\psi(S)\in\mathbb{R}$ as $\psi(S) = \sum_{i=1}^\infty a_i/10^i$ where $a_i = 0$ if $i\in S$ and $a_i = 1$ if $i\notin S$. (For example, if $S = \mathbb{N}\setminus\{2,4,5\}$ then $a_2 = a_4 = a_5 = 1$ and $a_i = 0$ for all other i. So $\psi(S) = 0.0101100\cdots = 0.01011$. Also, $\psi(\varnothing) = 0.111\cdots = 1/9$ and $\psi(\mathbb{N}) = 0.00\cdots = 0$.

In fact, the range of ψ is $[0, 1/9] \subset \mathbb{R}$.) Then ψ is one to one. Similarly, a one to one $\varphi : \mathcal{P}(D) \to L$ exists. So the composition $\varphi \circ \theta : \mathcal{P}(L) \to L$ is one to one. But the CONTRADICTS Theorem 7.8 (there is no one to one function from the power set of a set to the set itself; you might be familiar with this as Cantor's Theorem: The cardinality of the power set $\mathcal{P}(A)$ is strictly greater than the cardinality of the set A itself; $|\mathcal{P}(A)| > |A|$.) This contradiction shows that the assumption that $\mathbb{R}^2_\ell$ is normal is false and hence the Sorgenfrey plane $\mathbb{R}^2_\ell$ is not normal.

Note. As commented in the argument, we did not explicitly CONSTRUCT a set $A \subset L$ such that closed A and closed $L \setminus A$ violate the regularity of $\mathbb{R}^2_\ell$. However, as shown in Exercise 31.9, the set A of all elements of L with rational coefficients is such a set.

Revised: 8/30/2016