Section 32. Normal Spaces

Note. We give four conditions which each imply that a space is normal (one of them is metrizability). Two of the big results of this chapter, the Urysohn Metrization Theorem (Section 34) and the Tietze Extension Theorem (Section 35) apply to normal spaces.

Theorem 32.1. Every regular space with a countable basis is normal.

Note. The following result shows that every metrizable space is normal, so that we can extend our schematic of topological spaces from Section 31 as follows:

$$\text{metric spaces} \subset \text{Normal Spaces} \subset \text{Regular Spaces}$$

$$\subset \text{Hausdorff Spaces} \subset \text{Tychonoff Spaces}.$$

Theorem 32.2. Every metrizable space is normal.

Theorem 32.3. Every compact Hausdorff space is normal.

Theorem 32.4. Every well-ordered set X is normal in the order topology.

Note. The following two examples show that, in some sense, the normal spaces are not as well behaved as the regular and Hausdorff spaces (compare these examples to Theorem 31.2).

Example 1. In Exercise 32.9, you will show that for J uncountable, the product space $\mathbb{R}^J$ is not normal. Notice that $\mathbb{R}$ is regular (in fact, $\mathbb{R}$ is normal since it is a metric space; apply Theorem 32.2) and so by Theorem 31.2(b) $\mathbb{R}^J$ is regular. So this example shows that *the normal topological spaces are a proper subset of the regular topological spaces.*

Also notice that $\mathbb{R}$ is homeomorphic to the interval $(0,1)$ (a homeomorphism is given by $x \mapsto (2x-1)/(1-(2x-1)^2)$; this is the mapping $x \mapsto 2x-1$ which maps $(0,1)$ to $(-1,1)$ and $x \mapsto x/(1-x^2)$ which maps $(-1,1)$ to $\mathbb{R}$; see Example 5 from Section 18), so $\mathbb{R}^J$ is homeomorphic to $(0,1)^J$, and so $(0,1)^J$ is not normal. Now $[0,1]^J$ is a product of compact spaces and so is compact (we assume the Tychonoff Theorem of Section 37 for this claim). So $[0,1]^J$ is a compact Hausdorff (by Theorem 31,2(a)) space and therefore by Theorem 32.3 is normal. So this example shows that *a subspace* ($(0,1)^J$ here) *of a normal space* ($[0,1]^J$ here) *may not be normal.*

Since $\mathbb{R}^J$ is an uncountable product of normal space $\mathbb{R}$, this example shows that *an uncountable product of normal spaces may not be normal.* In the next example we will see that the product of two normal spaces may not be normal (and so the product of a countable number or even a finite number of normal spaces may not be normal).

Example 2. In this example, we consider the space $S_\Omega \times \overline{S}_\Omega$. Recall (see Lemma 10.2) that Ω is the largest element of a well-ordered set A where the section $S_\Omega = \{a \in A \mid a < \Omega\}$ is uncountable, but every other section S_α of A is uncountable. $\overline{S}_\Omega - S_\Omega \cup \{\Omega\}$. Let well-ordered $\overline{S}_\Omega$ have the order topology and then $\overline{S}_\Omega$ is normal by Theorem 32.4. Give S_Ω the subspace topology (which is the same as the order topology by Theorem 16.4) and so by Theorem 32.4, S_Ω is also normal. We will show that $S_\Omega \times \overline{S}_\Omega$ is not normal. This example then establishes the following (the first of which was established in Example 1):

1. *A regular space* ($S_\Omega \times \overline{S}_\Omega$ is regular since both S_Ω and $\overline{S}_\Omega$ are regular [being normal] by Theorem 31.2(b)) *may not be normal*, so that the set of normal spaces (again) is a proper subset of the set of regular spaces.

2. *A subspace* ($S_\Omega \times \overline{S}_\Omega$ here) *of a normal space* ($\overline{S}_\Omega \times \overline{S}_\Omega$) *may not be normal.* We now show that $\overline{S}_\Omega \times \overline{S}_\Omega$ is normal. Notice that $\overline{S}_\Omega$ satisfies the least upper bound property since any subset $A \subset \overline{S}_\Omega$ has Ω as an upper bound and the set $\{s \in \overline{S}_\Omega \mid a \leq s \text{ for all } a \in A\}$ has a least element since $\overline{S}_\Omega$ is well-ordered and this least element is the least upper bound of A. since $\overline{S}_\Omega$ is a closed interval ($\overline{S}_\Omega = [a, \Omega]$ where a is the least element of $\overline{S}_\Omega$), by Theorem 27.1, $\overline{S}_\Omega$ is compact. By Theorem 17.11, both $\overline{S}_\Omega$ is Hausdorff and the product $\overline{S}_\Omega \times \overline{S}_\Omega$ is Hausdorff. So $\overline{S}_\Omega \times \overline{S}_\Omega$ is a compact Hausdorff space and so is normal by Theorem 32.3.

3. *The product* ($\overline{S}_\Omega \times \overline{S}_\Omega$ here) *of two normal spaces* ($\overline{S}_\Omega$ here) *need not be normal.* This resolves the question at the end of Example 1 concerning products of normal space.

First, consider the "diagonal" $\Delta = \{(x,x) \mid x \in \overline{X}_\Omega\}$ in $\overline{S}_\Omega \times \overline{S}_\Omega$. Since $\overline{S}_\Omega$ is Hausdorff, Δ is closed in $\overline{S}_\Omega \times \overline{S}_\Omega$ for the following reason. For $x \neq y \in \overline{S}_\Omega \times \overline{S}_\Omega$ (so $(x, y \in \overline{S}_\Omega \times \overline{S}_\Omega \setminus \Delta)$, there are open disjoint U and V with $x \in U$ and $y \in V$. Then $U \times V$ is an open set containing (x, y) but $U \times V$ does not intersect Δ (because U and V share no element of $\overline{S}_\Omega$).

Therefore, in the subspace $S_\Omega \times \overline{S}_\Omega$, the set $A = \Delta \cap (S_\Omega \times \overline{S}_\Omega) = \Delta \setminus \{(\Omega, \Omega)\}$ is closed. Also the set $B = S_\Omega \times \{\Omega\}$ is closed in $S_\Omega \times \overline{S}_\Omega$ since S_Ω is closed in S_Ω and, since $\overline{S}_\Omega$ is Hausdorff by Theorem 17.11, $\{\Omega\}$ is closed by Theorem 17.8 (a product of closed sets is closed by Exercise 17.9 and Corollary 17.7). Sets A and B are disjoint (no element of A has second coordinate Ω while all elements of B have second coordinate Ω). ASSUME $S_\Omega \times \overline{S}_\Omega$ is regular and that there are disjoint open sets U and V of $S_\Omega \times \overline{S}_\Omega$ with $A \subset U$ and $B \subset V$. See Figures 32.2 and 32.3.

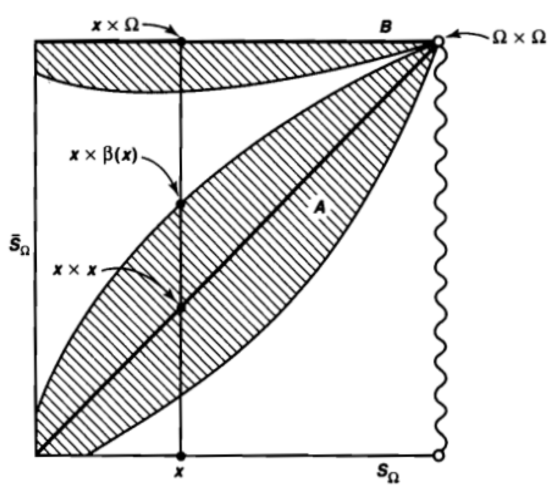

Figure 32.2

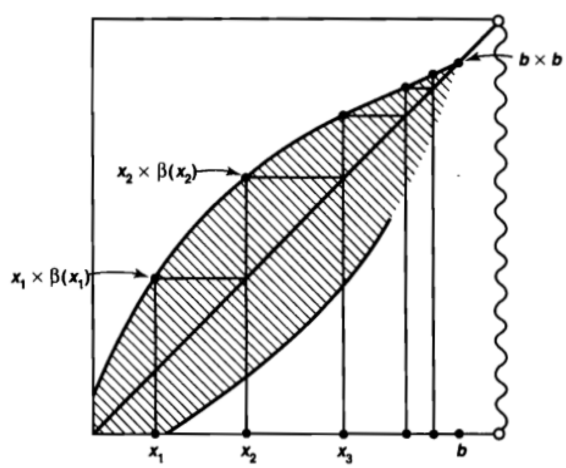

Figure 32.3

Given $x \in S_\Omega$, consider the "vertical" slice $\{x\} \times \overline{S}_\Omega$. Now if U contains all points of the form (x, β) for $x < \beta < \Omega$, then the "top" point (x, Ω) would be a limit point of U, which it is not since V is an open set containing (x, Ω) which is disjoint from U. So there is some β with $x < \beta < \Omega$ such that $(x, \beta) \notin U$. For the set of all such $\beta \in S_\Omega$, let $\beta(x)$ denote the least such β (which can be done since S_Ω is well-ordered).

Define a sequence of points of S_Ω as follows. Let X_1 be any point of S_Ω. let $x_2 = \beta(x_1)$ and in general (recursively, that is) let $x_{n+1} = \beta(x_n)$ (see Figure 32.3). Since $\beta(x) > x$ for all $x \in S_\Omega$, the sequence satisfies $x_1 < x_2 < \cdots$. The set $\{x_n\}$ is countable and therefore has an upper bound in S_Ω (see Theorem 10.3). Let $b \in S_\Omega$ be the least upper bounds of set $\{x_n\}$ (which exists since the set of upper bounds of set $\{x_n\}$ has a least upper bound because S_Ω is well-ordered). Because the sequence is increasing, it must converge to its least upper bound (if not, there is a smaller least upper bound). That is, $x_n \to b$. Since $\beta(x_n) = x_{n+1}$ then $\beta(x_n) \to b$. So the sequence $(x_n, \beta(x_n))$ in $S_\Omega \times \overline{S}_\Omega$ converges: $(x_n, \beta(x_n)) \to (b, b)$ in $S_\Omega \times \overline{S}_\Omega$. But $(b, b) \in \Delta \subset A \subset U$. But U contains no points of the sequence $(x_n, \beta(x_n))$ (by the choice of the $\beta(x_n)$). So by the definition of limit of a sequence, open set U containing (b, b) and containing no terms of the sequence shows that the limit of sequence $(x_n, \beta(x_n))$ cannot be (b, b), a CONTRADICTION. So the assumption that $S_\Omega \times \overline{S}_\Omega$ is regular is false and hence $S_\Omega \times \overline{S}_\Omega$ is not regular.

Note. In summary, we have the following properties of Hausdorff, regular, and normal spaces in terms of subset inclusions:

$$\text{(normal spaces)} \subsetneq \text{(regular spaces)} \subsetneq \text{(Hausdorff spaces)}.$$

Examples showing the proper inclusions are:

1. Hausdorff by not regular: $\mathbb{R}_K$ ($\mathbb{R}$ under the K-topology; Example 1 of Section 31).

2. Regular but not normal:

- $\mathbb{R}^J$ where J is uncountable (Example 1 of Section 32).
- $S_\Omega \times \overline{S}_\Omega$ (Example 2 of Section 32).

In terms of subspaces and products we have:

Space	**Subspace/Reason**	**Product/Reason**
Hausdorff	Hausdorff/Theorem 31.2(a)	Hausdorff/Theorem 31.2(a)
Regular	Regular/Theorem 31.2(b)	Regular/Theorem 31.2(b)
Normal	Maybe not normal/$(0,1)^J \subset [0,1]^J$, J uncountable (Example 1, §32)	Maybe not normal/$\mathbb{R}_\ell \times \mathbb{R}_\ell$, (Examples 1, 2, §31)
	Maybe not normal/$S_\Omega \times \overline{S}_\Omega \subset \overline{S}_\Omega \times \overline{S}_\Omega$ (Example 2, §32)	Maybe not normal/$\mathbb{R}^J$, J uncountable, (Example 1, §32)

Revised: 8/20/2016