Section 33. The Urysohn Lemma

Note. Munkres declares the Urysohn Lemma "the first deep theorem of the book." It will be used to prove the Urysohn Metrization Theorem (in Section 34), the Tietze Extension Theorem (in Section 35), and an embedding theorem for manifolds (in Section 36). The Urysohn Lemma states that in a normal space X, for given closed disjoint set A and B there is a continuous real valued function from X to $[a, b] \subset \mathbb{R}$ such that $f(x) = 1$ for all $x \in A$ and $f(x) = b$ for all $x \in B$. Think about it like this: In a regular space, there is some "room" between two open sets. In this room, function f increases from a to b between set A and set B.

Theorem 33.1. The Urysohn Lemma.
Let X be a normal space. Let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map $f : X \to [a, b]$ such that $f(x) = a$ for every $x \in A$, and $f(x) = b$ for every $x \in B$.

Note. We now use the existence of the function f of the Urysohn Lemma to define a new type of separation of sets.

Definition. If A and B are two subsets of topological space X and if there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, then sets A and B can be *separated by a continuous function.*

Note. The Urysohn Lemma shows that any two disjoint closed sets in a normal space can be separated by a continuous function.

Note. We now define a new separation axiom for spaces satisfying a condition of regularity along with a type of separation by a continuous function, but not necessarily satisfying the condition of normality.

Definition. A topological space X is *completely regular* if one-point sets are closed in X and if for each $x_0 \in X$ and each closed $A \subset X$ not containing x_0, there is a continuous $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

Note. By the Urysohn Lemma, a normal space is completely regular. If X is completely regular with given f, then $U = f^{-1}([0,1/2))$ and $V = f^{-1}((1/2,1])$ are open disjoint sets in X with $x_0 \in U$ and $A \subset V$ and hence X is regular. So schematically we have:

$$\text{(normal spaces)} \subset \text{(completely regular spaces)} \subset \text{(regular spaces)}.$$

In Example 1 and Exercise 33.11, these inclusions are shown to be proper. Munkres mentions that completely regular spaces are sometimes ("facetiously") called $T_{3\frac{1}{2}}$ spaces since they lie between T_3 (regular) spaces and T_4 (normal) spaces.

Note. We saw by example in Section 32 that normal spaces are not as well-behaved as the other spaces in terms of subspaces and products. However, completely regular spaces are well-behaved, as shown in the next theorem.

Theorem 33.2. A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular.

Example 1. Since $\mathbb{R}_\ell$, S_Ω, and $\overline{S}_\Omega$ are normal (by Example 2 of Section 31, Theorem 32.4, and Theorem 32.4, respectively) and so are complete regular. however, neither $\mathbb{R}_\ell^2 = \mathbb{R}_\ell \times \mathbb{R}_\ell$ nor $S_\Omega \times \overline{S}_\Omega$ are normal (by Example 3 of Section 31 and Example 2 of Section 32, respectively). So the set of completely regular spaces is a proper subset of the set of normal spaces.

Note. In Exercise 33.11 you will show that there is an example of a regular space which is not completely regular, showing that the regular spaces are a proper subset of the completely regular spaces. The example is due to John Thomas in "A Regular Space, Not Completely Regular," *American Mathematical Monthly*, **76**: 181–182 (1969).

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