Section 34. The Urysohn Metrization Theorem

Note. We know several properties of metric spaces (see Sections 20, 21, and 28, for example). We have shown certain spaces are *not* metrizable by showing that they violate properties of metric spaces. See Example 1 of Section 21 ($\mathbb{R}^\omega$ is the box topology is not metrizable), Example 2 of Section 21 (as uncountable product of $\mathbb{R}$ is not metrizable), and Example 3 of Section 28 (S_ω is not metrizable). In this section we give conditions under which a space *is* metrizable. In fact, in Section 40 we give necessary and sufficient conditions for a topological space to be metrizable in the Nagata-Smirnov Metrization Theorem (Theorem 40.3). We give two proofs of the Urysohn Metrization Theorem, each has useful generalizations which we will use later.

Note. We modify the order of the proof from Munkres' version by first presenting a lemma.

Lemma 34.A. If X is a regular space with a countable basis, then there exists a countable collection of continuous functions $f_m : X \to [0.1]$ having the property that given any point $x_0 \in X$ and any neighborhood U of x_0, there exists an index n such that $f_n(x_0) > 0$ and $f_n(x) = 0$ for all $x \in U$.

Theorem 34.1. The Urysohn Metrization Theorem.

Every regular space X with a countable basis is metrizable.

Note. Notice that each of the two proofs of the Urysohn Metrization Theorem depend on showing that $F : X \to \mathbb{R}^\omega$ as $F(x) = (f_1(x), f_2(x), \ldots)$ where the f_n are as given in Lemma 34.A is a homeomorphism between X and either $[0,1]^\omega$ or a subspace of $[0,1]^\omega$, and arguing that $\mathbb{R}^\omega$ is a metric space. We can follow the first proof (except for the metrizability part) to prove the following result concerning embeddings of spaces (not necessarily regular) in $\mathbb{R}^J$.

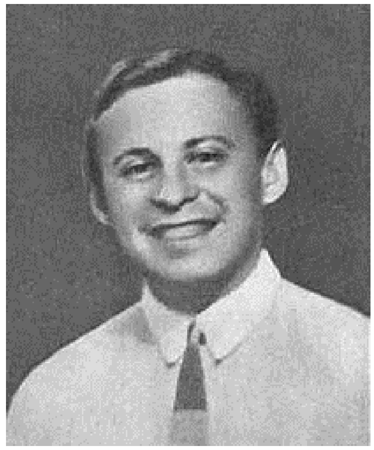
Theorem 34.2. Embedding Theorem.
Let X be a space in which one-point sets are closed. Suppose $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of continuous functions $f_\alpha : X \to \mathbb{R}$ satisfying the requirement that for each $x_0 \in X$ and each neighborhood U of x_0, there is an index α such that $f_\alpha(x_0) > 0$ and $f_\alpha(x) = 0$ for $x \in X \setminus U$. Then the function $F : X \to \mathbb{R}^J$ defined as $F(x) = (f_\alpha(x))_{\alpha \in J}$ is an embedding of X in $\mathbb{R}^J$. If $f_\alpha : X \to [0,1]$ for each $\alpha \in J$, then F embeds X in $[0,1]^J$.

Note. The proof of Theorem 34.2 is the same as the first proof of the Urysohn Metrization Theorem, except that $n \in \mathbb{N}$ is replaced by $\alpha \in J$ and $\mathbb{R}^\omega$ is replaced with $\mathbb{R}^J$ (and the metrization claims are dropped). We need the explicit claim that one-point sets are closed to insure the existence of some $\alpha \in J$ with $x \neq y$ and $f(x) \neq f(y)$ (for $x \neq y$, consider the neighborhood $X \setminus \{y\}$ od x and the neighborhood $X \setminus \{x\}$ of y). You will check all the details of the proof in Exercise 34.6.

Definition. Let X be a topological space. A family of functions $\{f_\alpha\}_{\alpha \in J}$ satisfying the conditions given in the Embedding Theorem (Theorem 34.2) *separates points from closed sets.*

Theorem 34.3. A space X is completely regular if and only if it is homeomorphic to a subspace of $[0, 1]^J$ for some indexing set J.

Note. Pavel Urysohn entered the University of Moscow in 1915 where he attended lectures by Nikolai Luzin (1883–1950) and Dimitri Egorov (1869–1931). He graduated with his doctorate in 1921 and became an assistant professor at the University of Moscow. With direction from Egorov, Urysohn studied topology and started the area of dimension theory. He visited Göttingen, Germany in 1923 and met with David Hilbert (1862-1943) and L. E. J. Brouwer (1881-1966). In 1924, he traveled with colleague Pavel Aleksandrov (1896–1982) through western Europe, met, and corresponded with Felix Hausdorff (1868–1942). While on this trip, Urysohn drowned while out for a swim. "Urysohn's main contributions, in addition to the theory of dimension discussed above, are the introduction and investigation of a class of normal surfaces, metrization theorems, and an important existence theorem concerning mapping an arbitrary normed space into a Hilbert space with countable basis." This quote and these historical notes are based on the MacTutor History of Mathematics archive website: `www-history.mcs.st-and.ac.uk/Biographies/Urysohn.html`



Pavel Urysohn; February 3, 1898–August 17, 1924

Image from https://en.wikipedia.org/wiki/Pavel_Urysohn

Revised: 8/29/2016