Section 36. Embeddings of Manifolds

Note. In this section we give a definition of a "manifold" which is, on the surface of it(!), rather different from the definition encountered in a differential geometry class. We show that every compact m-manifold ("m-dimensional manifold") can be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$.

Definition. An m-manifold is a Hausdorff space with a countable basis such that each $x \in X$ has a neighborhood that is homeomorphic with an open subset of $\mathbb{R}^m$. A 4-manifold is a *curve* and a 2-mainfold is a *surface*.

Note. In differential geometry, a manifold is defined as in terms of mappings between the manifold and $\mathbb{R}^m$. where two mappings overlap on the manifold, a level of differentiability is required of the composition of these mappings. See my notes from Differential Geometry (MATH 5310): `http://faculty.etsu.edu/gardnerr/5310/5310pdf/dg1-9.pdf` and `http://faculty.etsu.edu/gardnerr/5310/5310pdf/Waldrel-2-1.pdf`. In this section, we are interested in the dimension of a manifold and we do not address differentiability ("smoothness") so we don't consider the mapping which are of interest from a differential geometry standpoint.

Definition. If $\varphi : X \to \mathbb{R}$, the *support* of φ is the closure of the set $\varphi^{-1}(\mathbb{R} \setminus \{0\})$.

Example. If $\varphi : \mathbb{R} \to \mathbb{R}$ defined as

$$\varphi(x) = \begin{cases} x+1 & \text{if } x \in (-\infty, -1) \\ 0 & \text{if } x \in [-1, 1] \\ x-1 & \text{if } x \in (1, \infty) \end{cases}$$

then the support of φ is $(-\infty, -1] \cup [1, \infty)$. Notice (in general) that if x lies outside the support of φ then there is a neighborhood of x on which φ is zero.

Definition. let $\{U_1, U_2, \ldots, U_n\}$ be a finite open covering of space X. An indexed family of continuous functions $\varphi_i : X \to [0, 1]$ for $i = 1, 2, \ldots, n$ is a *partition of unity dominated by* $\{U_i\}$ if

(1) (support φ_i) $\subset U_i$ for $i = 1, 2, \ldots, n$, and

(2) $\sum_{i=1}^{n} \varphi_i(x) = 1$ for each $x \in X$.

Theorem 36.1. Existence of Finite Partitions of Unity.
Let $\{U_1, U_2, \ldots, U_n\}$ be a finite open covering of the normal space X. Then there exists a partition of unity dominated by $\{U_i\}$.

Note. Now for the main result of this section.

Theorem 36.2. If X is a compact m-manifold then X can be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$.

Note. The compactness of X leads us to parameter n, so we don't really know how big n might be. In Chapter 8 it is shown that every compact m-manifold can be embedded in $\mathbb{R}^{2m+1}$ (see Corollary 50.8).

Note. This section completes the "irreducible core" of of topology (see Munkres, page xii). You now have the background to cover any of the remaining 4 chapters of Part I (General Topology), or Chapter 9 of Part II (Algebraic Topology).

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