Chapter 5. The Tychonoff Theorem

Section 37. The Tychonoff Theorem

Note. In Theorem 26.7 we say that a finite product of compact spaces is compact. In this section we prove that an arbitrary product of compact spaces is compact. The proof Munkres gives uses Zorn's Lemma, but Exercise 37.5 outlines a proof based on the Well-Ordering Theorem and transfinite induction (see Exercise 10.7 for a statement of the Principle of Transfinite Induction).

Note. In the proof that $X \times Y$ is compact for compact X and Y (Theorem 26.7), we used the Tube Lemma (Lemma 26.8) to cover slices, $\{x_0\} \times Y$, with tubes, $W \times Y$ where W is a neighborhood of x_0, and the compactness of X and Y to find finitely many tubes covering $X \times Y$, which lead to finitely many elements of $\mathcal{A}$ covering $X \times Y$ where $\mathcal{A}$ was a given open covering of $X \times Y$. Munkres outlines the ideas of the proof for a general product of compact spaces on pages 231 and 232 in three steps. First, start with a collection of sets $\mathcal{A}$ that has the finite intersection property. Second, expand the collection $\mathcal{A}$, retaining the finite intersection property, to a new collection $\mathcal{D}$. Third, $\mathcal{D}$ is made as large as possible, using Zorn's Lemma. The third step (and the "large as possible" property) is given in the next lemma. The use of Zorn's Lemma is restricted to the following lemma, so if we are pressed for time, we can skip the proof of Lemma 37.1 and just cover the proofs of Lemma 37.2 and the Tychonoff Theorem.

Lemma 37.1. Let X be a set. Let $\mathcal{A}$ be a set ("collection") of subsets of X having the finite intersection property. Then there is a collection $\mathcal{D}$ of subsets of X such that $\mathcal{D}$ contains $\mathcal{A}$ and $\mathcal{D}$ has the finite intersection property, and no collection of subsets of X that properly contains $\mathcal{D}$ has this property. Such a collection $\mathcal{D}$ is said to be *maximal* with respect to the finite intersection property.

Note. The next lemma gives some properties of the maximal collection $\mathcal{D}$ of the previous lemma.

Lemma 37.2. Let X be a set. Let $\mathcal{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then:

(a) Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.

(b) If A is a subset of X that intersects every element of $\mathcal{D}$, then A is an element of $\mathcal{D}$.

Theorem 37.3. Tychonoff Theorem.
An arbitrary product of compact spaces is compact in the product topology.

Note. Andrey Nikolayevih Tikhonov (or "Andrei Nikolaevich Tikhonov"... or "Tychonoff") (1906–1993) attended the Moscow University starting in 1922 and published his first paper in 1925 while still an undergraduate. His first studies (related to those of Aleksandrov and Urysohn) were on conditions for a topological space to be metrizable. By 1926 he had defined the product which we saw in Section 19 and which is sometimes called the "Tychonoff topology." His definition allowed him to prove results concerning the product of topological spaces, such as the one we explored in this section. He has reputation was established before he graduated in 1927. In that year, he became a research student at Moscow University. He became a professor at Moscow University in 1936. He went on to study functional analysis, ODEs and PDEs, computational mathematics, and mathematical physics. He received numerous awards, was a member of the USSR Academy of Sciences, and deputy director of the Institute of Applied Mathematics of the USSR Academy of Sciences for many years. These historical notes and the photo below are from `http://www-groups.dcs.st-and.ac.uk/history/Biographies/Tikhonov.html`.

Andrey Tychonoff

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