Chapter 6. Metrization Theorems and Paracompactness

Note. Recall that the Urysohn Metrization Theorem states that every regular space X with a countable basis is metrizable. By Theorem 32.2, every metrizable space is normal, so the condition of normality is a necessary one. However, the condition of a countable basis is not a necessary condition for metrizability. In this chapter, we give two necessary and sufficient conditions for a topological space to be metrizable. The first is based on local finiteness (see Section 39) and is called the Nagata-Smirnov Metrization Theorem (see Section 40). The second is based on paracompactness (see Section 41) and is called the Smirnov Metrization Theorem (see Section 42).

Section 39. Local Finiteness

Note. In this section, we define "locally finite" and "countably local finite" topological spaces. We prove a lemma concerning metrizable spaces and countable local finiteness which is used in the proof of the Nagata-Smirnov Theorem in the next section.

Definition. Let X be a topological space. A collection $\mathcal{A}$ of subsets of X is *locally finite* in X if every point of X has a neighborhood that intersects only finitely many elements of $\mathcal{A}$.

Example 1. The collection of intervals $\mathcal{A} = \{(n, n+2) \mid n \in \mathbb{Z}\}$ is locally finite in $\mathbb{R}$. For any $n \in \mathbb{Z}$, there is a sufficiently small neighborhood of n which intersects only three elements of $\mathcal{A}$ (namely, $(n-2, n)$, $(n-1.n+1)$, and $(n, n+2)$). For any $x \notin \mathbb{Z}$, there is a sufficiently small neighborhood of x which intersects only two elements of $\mathcal{A}$. The collection of intervals $\mathcal{B} = \{(0, a/n) \mid n \in \mathbb{N}\}$ is locally finite in $X = (0,1)$ but not in $X = \mathbb{R}$ (since any neighborhood of 0 in $\mathbb{R}$ intersects all elements of $\mathcal{B}$). Similarly, $\mathcal{C} = \{(1/(n+1), 1/n) \mid n \in \mathbb{N}\}$ is locally finite in $(0,1)$ but not in $\mathbb{R}$ (for the same reason as above); notice that the elements of $\mathcal{C}$ are disjoint.

Lemma 39.1. Let $\mathcal{A}$ be a locally finite collection of subsets of X. Then:

(a) Any subcollection of $\mathcal{A}$ is locally finite.

(b) The collection $\mathcal{B} = \{\overline{A}\}_{A \in \mathcal{A}}$ of the closures of the elements of $\mathcal{A}$ is locally finite.

(c) $\overline{\cup_{A \in \mathcal{A}} A} = \cup_{A \in \mathcal{A}} \overline{A}$.

Note. The following two definitions are built on the idea of a locally finite collection of sets.

Definition. A collection $\mathcal{B}$ of subsets of X is *countably locally finite* if $\mathcal{B}$ can be written as the countable union of collections $\mathcal{B}_n$, each of which is locally finite. This is sometimes called "σ-locally finite."

Definition. Let $\mathcal{A}$ be a collection of subsets of X. A collection $\mathcal{B}$ of subsets of X is a *refinement* of $\mathcal{A}$ if for each $B \in \mathcal{B}$ there is $A \in \mathcal{A}$ with $B \subset A$. If the elements of $\mathcal{B}$ are open sets, $\mathcal{B}$ is an *open refinement* of $\mathcal{A}$; if they are closed sets, $\mathcal{B}$ is a *closed refinement* of $\mathcal{A}$.

Note. We will use the following in the proof of the Nagata-Smirnov Metrization Theorem (Theorem 40.3).

Lemma 39.2. Let X be a metrizable space. If $\mathcal{A}$ is an open covering of X, then there is an open covering $\mathcal{E}$ of X refining $\mathcal{A}$ that is countable locally finite.

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