Section 40. The Nagata-Smirnov Metrization Theorem

Note. In this section, we show that the metrizability of a topological space is equivalent to the space being regular and having a countably locally finite basis for the topology.

Note. Similar to the second proof of the Urysohn Metrization Theorem (Theorem 34.1), we embed the space in a metric space. This time, the metric space is the product space $(\mathbb{R}^J, \overline{\rho})$ for some index set J, where $\overline{\rho}$ is the uniform metric (see Section 20).

Definition. A subset A of a space X is a G_δ *set* in X if A equals the intersection of a countable collection of open subsets of X.

Note. Notice that if A is a G_δ set in space X, say $A = \cap_{n=1}^{\infty} U_n$ where each U_n is open, then by De Morgan's Law (see page 11)

$$A^c = X \setminus A = X \setminus (\cap_{n=1}^{\infty} U_n) = X \cap (\cap_{n=1}^{\infty} U_n)^c = X \cap (\cup_{n=1}^{\infty} U_n^c) = \cup_{n=1}^{\infty} U_n^c,$$

and so A^c is a countable union of closed sets. Such a set is called an "F_σ set" (see Exercise 40.2). In fact, we can create other types of sets by iterating the operations of countable intersections and countable unions. For example, we can define a set as "$G_{\delta\sigma}$" is if is a countable union of G_δ sets, and we can define a set as "$F_{\sigma\delta}$" if it is a countable intersection of F_σ sets. Similarly we can define sets as $G_{\delta\sigma\delta}, G_{\delta\sigma\delta\sigma}, \ldots$ and $F_{\sigma\delta\sigma}, F_{\sigma\delta\sigma\delta}, \ldots$ (the pattern here is to use a σ when considering countable unions and a δ when considering countable intersections). Notice that this gives us a way to describe what certain sets "look like." For example, a $G_{\delta\sigma\delta}$ set is a countable intersection of countable unions of countable intersections of open sets! We can also address the topic of a σ-algebra $\mathcal{A}$ (which is, by definition, closed under the operations of countable unions, countable intersections, and complements) and the collection of *Borel sets*. The collection of Borel sets is the smallest σ-algebra containing the open sets. For more details on these ideas in the setting of $\mathbb{R}$, see my class notes for Real Analysis 1 (MATH 5210): `http://faculty.etsu.edu/gardnerr/5210/notes/1-4.pdf`. In Real Analysis 1, we define the Lebesgue measure of certain sets of real numbers. The collection of Lebesgue measurable sets form a σ-algebra which contains the Borel sets.

Example. In $\mathbb{R}$ (under the usual topology) notice that every open interval (a, b) is F_σ because

$$\bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right] = (a, b).$$

Similarly, unbounded intervals $(-\infty, b)$ and (a, ∞) are F_σ because

$$(-\infty, b) = \bigcup_{n=1}^{\infty} \left[-n, b - \frac{1}{n}\right] \text{ and } (a, \infty) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, n\right].$$

Also, $\mathbb{R} = (-\infty, \infty)$ is F_σ since it is a closed set. Recall that any open set of real numbers is a countable disjoint union of open intervals (see the first example in the notes for Section 13). Since a countable union of countable sets is countable (by Theorem 7.5), it follows that every open set of real numbers is F_σ Since the complement of an F_σ set is a G_δ set, every closed set of real numbers is a G_δ set.

Note. We need two lemmas for the proof of the Nagata-Smirnov Metrization Theorem.

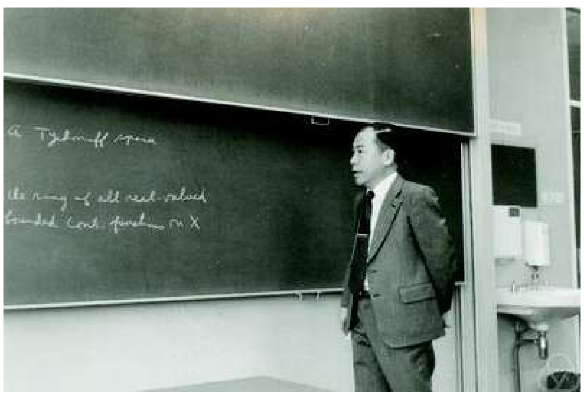
Lemma 40.1. Let X be a regular space with a basis $\mathcal{B}$ that is countably localy finite. Then X is normal, and every closed set in X is a G_δ set in X.

Lemma 40.2. Let X be normal. Let A be a closed G_δ set. Then there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \not\in A$.

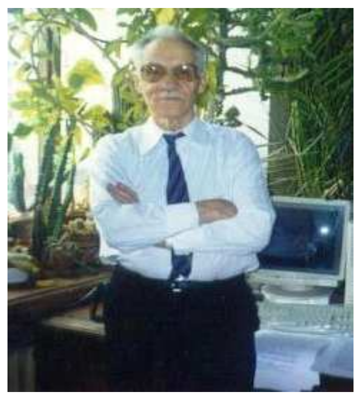
Note. We now have the equipment to give necessary and sufficient conditions for the metrizability of a topological space.

Theorem 40.3. **The Nagata-Smirnov Metrization Theorem.**
A topological space X is metrizable if and only if X is regular and has a basis that is countably locally finite.

Note. Sometimes this result is called the Nagata-Smirnov-Bing Theorem, since it was independently proved by three individuals. The first to publish was Jun-iti Nagata (1925–November 6, 2007) in: On a Necessary and Sufficient Condition for Metrizability, *Journal of the Institute of Polytechnics, Osaka City University. Series A, Mathematics* **1**(2), (1950), 93–100. At the time, this Japanese journal may not have been widely circulated in Europe. In 1951, Yurii Mikhailovich Smirnov (September 19, 1921–September 3, 2007) published a proof in Russian: A Necessary and Sufficient Condition for Metrizability of a Topological Space, *Dokl. Akad. SSSR* (*Proceedings of the USSR Academy of Sciences*) **77**, (1951), 197–200. Also in 1951, R. H. Bing (October 20, 1914–April 28, 1986) gave a proof in: Metrization of Topological Spaces, *Canadian Journal of Mathematics* **3**, (1951) 175–186. Given this history, it seems that title "The Nagata-Smirnov-Bing Theorem" is appropriate!



Nagata (1925–2007)



Smirnov (1921–2007)

Bing (1914–1986)

Images from Wikipedia, `http://at.yorku.ca/t/o/p/d/06.htm`, and the MacTutor History of Mathematics archive; accessed 10/1/2016.

Nagata was Japanese, Smirnov was Russian, and Bing was from born and died in Texas. Bing served as president of the Mathematical Association of America in 1963 and 1964, served as president of the American Mathematical Society in 1977 and 1978, was department chair at University of Wisconsin, Madison from 1958 to 1960, and was department chair at University of Texas at Austin from 1975 to 1977 (see his Wikipedia biography or his MacTutor biography).

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