Section 41. Paracompactness

Note. In this section, we define "paracompactness" and see that the definition involves locally finite open covers. The class of paracompact topological spaces includes the compact spaces (by definition) and the metrizable spaces (by Theorem 41.4).

Note. Recall that a topological space X is compact if every open covering has a finite subcovering. Notice that for a given open covering $\mathcal{A}$ of X, if $\mathcal{B}$ is a finite open refinement of $\mathcal{A}$ that covers X, then each element of $\mathcal{B}$ is a subset of an element of $\mathcal{A}$ and so $\mathcal{A}$ has a finite subcovering of X. Of course $\mathcal{A}$ is a refinement of itself, so we can say: "A topological space X is compact if and only if every open covering of X has a finite open refinement $\mathcal{B}$ that covers X."

Definition. A topological space X is *paracompact* if every open covering of $\mathcal{A}$ has a locally finite open refinement $\mathcal{B}$ of X.

Note. Since a finite open covering is trivially a locally finite open covering, then a compact topological space is a paracompact space.

Note. Munkres comments that Bourbaki includes as part of the definition of paracompact the condition of Hausdorff. Munkres does not make this assumption.

Note. "Nicolas Bourbaki" is the name adopted by a group of French mathematicians in 1935. They chose the name in honor of a French general who played an important role in the Franco-Prussian War of 1870–71. The group was formed initially to write an analysis textbook. They wanted a book modeled after Euclid's *Elements of Geometry*. The main goal was to standardize mathematical terminology and to maintain the highest possible level of rigor. They have since published books on set theory, algebra, topology, functions of one real variable, topological vector spaces, integration, commutative algebra, Lie groups and Lie algebras. The ETSU Sherrod Library has Bourbaki's *Theory of Sets* (QA248.B73413). For more information, see Maurice Mashaal's *Bourbaki—A Secret Society of Mathematicians* (translated from French by Anna Pierrehumbert), American Mathematical Society (2006).

Example 1. We now show that $\mathbb{R}^n$ is paracompact (but $\mathbb{R}^n$ is not compact, of course). Let $X = \mathbb{R}^n$ and let $\mathcal{A}$ be an open covering of X. We now construct a locally finite open refinement $\mathcal{C}$ of $\mathcal{A}$ that covers X. First, we define a collection of pen balls. Let $B_0 = \varnothing$ and for each $n \in \mathbb{N}$ let B_m denote the ball of radius m centered at $\mathbf{0}$ (that is, $B_m = B(\mathbf{0}, m)$). Given m, set $\overline{B}_m$ is compact in $\mathbb{R}^n$ by the Heine-Borel Theorem (Theorem 27.3) so choose finitely many elements of $\mathcal{A}$ that cover $\overline{B}_m$ and intersect each one with the open set $X \setminus \overline{B}_{m-1}$, and let $\mathcal{C}_m$ denote this collection of open sets (each an open subset of an element of $\mathcal{A}$). So $\mathcal{C} = \cup_{m=0}^{\infty} \mathcal{C}_m$ is an open refinement of $\mathcal{A}$. Not $\mathcal{C}$ covers X since for any $\mathbf{x} \in X$, there is some smallest $m \in \mathbb{N}$ such that $\mathbf{x} \in \overline{B}_m$ (namely, some m where $\|\mathbf{x}\| \le m < \|\mathbf{x}\| + 1$) and so $\mathbf{x}$ is an element of $\mathcal{C}_m$. Now collection $\mathcal{C}$ is locally finite since for given

$\mathbf{x} \in X$, neighborhood B_m intersects only finitely many elements of $\mathcal{C}$, namely those elements in collection $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_m$. So $\mathcal{C}$ is a locally finite open refinement of $\mathcal{A}$ that covers X and hence X is paracompact.

Note. Paracompact spaces share some properties with compact spaces, as illustrated in the results of this section.

Theorem 41.1. Every paracompact Hausdorff space X is normal.

Theorem 41.2. Every closed subspace of a paracompact space is paracompact.

Note. The following two examples show that subspaces of paracompact spaces need not have certain properties.

Example 2. We claim that a paracompact subspace Y of a Hausdorff space X need not be closed in X. Let $X = \mathbb{R}$ and $= (0, 1)$. Then X is Hausdorff and Y is paracompact by Example 1 (since $Y = (0, 1)$ is homeomorphic to $\mathbb{R}$), but $Y = (0, 1)$ is not closed in $X = \mathbb{R}$.

Example 3. We claim (unsurprisingly, perhaps) that a subspace Y of a paracompact space X need not be paracompact. Consider the space $X = \overline{S}_\Omega \times \overline{S}_\Omega$. Now $\overline{S}_\Omega$ is compact (in fact, $\overline{S}_\Omega$ is homeomorphic to the one-point compactification of S_Ω by Exercise 29.7), so $\overline{S}_\Omega \times \overline{S}_\Omega$ is compact by Theorem 26.7. Hence it is also paracompact. The subspace $Y = S_\Omega \times \overline{S}_\Omega$ is Hausdorff (see Example 2 of Section 32 and Theorem 17.11) but it is not normal by Example 2 of Section 32. So by Theorem 41.1, Y cannot be paracompact (otherwise, Y would be normal).

Note. The following is a major step in our proof that every metrizable space is paracompact (Theorem 41.4). The result is due to Ernest Michael (1925–2013) and originally appeared in "A Note on Paracompact Spaces," *Proceedings of the American Mathematical Society* **4** (1953), 831–838. Notice that condition (4) of Lemma 41.3 is the condition of paracompactness. Michael shows that an F_σ subset of a paracompact space is paracompact (in contrast to the previous example and generalizing Lemma 41.2). Although he does not seem to address metrizability, he does give a couple of results on products of spaces. You can likely find this paper online in the public domain. It seems that Michael (and others) attributes the introduction of the idea of paracampactness to Jean Dieudonné (1906–1992) in "Une généralization des espaces compacts," *Journal de Mathématiques Pures et Appliquées* **23** (1944), 65-76. Dieudonné seems to have given first proof of Theorem 41.1 (possibly in his 1944 paper), as it is referred to as a "Theorem of Dieudonné" on Wikipedia (as of 10/26/2016; see the article on "Paracompact space"). By the way, Dieudonné was one of the founding members of the Bourbaki group.

Lemma 41.3. Let X be a regular topological space. The following conditions on X are equivalent. Every open covering of X has a refinement that is:

(1) an open covering of X and countably locally finite,

(2) a covering of X and locally finite,

(3) a closed covering of X and locally finite, and

(4) an open covering of X and locally finite (that is, X is paracompact).

Note. The following two results give categories of topological spaces which are paracompact. The first theorem will be used in the next section in the proof of the Smirnov Metrization Theorem (Theorem 42.1). The next theorem is due to Arthur H. Stone (1916–2000) and appears in his "Paracompactness and Product Spaces," *Bulletin of the American Mathematical Society* **54** (1948), 977-982.

Theorem 41.4. Every metrizable space is paracompact.

Theorem 41.5. Every regular Lindelöf space is paracompact.

Note. Recall that the product of compact spaces is compact (for a finite product by Theorem 26.7 and for an arbitrary product by the Tychnoff Theorem, Theorem 37.3). The following example shows that none of this necessarily holds for paracompact sets and is due to R. H. Sorgenfrey and appeared in 1947 as "On the Topological Product of Paracompact Spaces," *Bulletin of the American Mathematical Society* **53**, (1947), 631–632. This paper seems to have introduced $\mathbb{R}_\ell^2$, the Sorgenfrey plane.

Example 4. We show that a product of two paracompact spaces may not be paracomapct. Recall that $\mathbb{R}_\ell$ denotes the lower limit topology on $\mathbb{R}$ generated by $\{[a,b) \mid a,b \in \mathbb{R}, a < b\}$. $\mathbb{R}_\ell$ is Lindelöf as shown in Example 3 of Section 30. $\mathbb{R}_\ell$ is normal be Example 2 of Section 31. So by Theorem 41.5, $\mathbb{R}_\ell$ is paracompact. Now $\mathbb{R}_\ell = \mathbb{R}_\ell \times \mathbb{R}_\ell$ is Hausdorff because $\mathbb{R}_\ell$ is Hausdorff (for $x \neq y$, say $x \leq y$, consider $[x-1, x+(x+y)/2)$ and $[x+(x+y)/2, y+1))$ so $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is Hausdorff by Theorem 17.11. But $\mathbb{R}_\ell^2$ is not normal as shown in Example 3 of Section 31. So $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not paracompact by Theorem 17.11.

Example 6. We claim that $\mathbb{R}^J$ (under the product topology) is not paracompact if J is uncountable. Now $\mathbb{R}^J$ is not normal by Example 1 of Section 32 (really, by Exercise 32.9). $\mathbb{R}$ is Hausdorff so $\mathbb{R}^J$ is Hausdorff by Theorem 31.2(a). So $\mathbb{R}^J$ is not paracompact by Theorem 41.1.

Note. We now have the equipment to prove the Smirnov Metrization Theorem (Theorem 42.1 of the next section), so if we are low on time we can skip the rest of this section.

Note. We now return to the concept of a partition of unity which was introduced in Section 36 (where the concept was used in proving that a compact m-manifold can be embedded in $\mathbb{R}^N$, Theorem 36.2).

Note. Recall that the *support* of $\varphi : X \to \mathbb{R}$ is the closure of the set $\varphi^{-1}(\mathbb{R} \setminus \{0\})$. The following definition of "partition of unity" is in the setting of arbitrary open coverings (instead of finite open coverings as in Section 36) and requires local finiteness of the indexed family $\{\text{Support}(\varphi_\alpha\}_{\alpha \in J}$ so that the sum $\sum_{\alpha \in J} \varphi_\alpha(x)$ only involves finitely many nonzero terms for any given x (so the sum makes sense even if J is uncountable).

Definition. Let $\{U_\alpha\}_{\alpha \in J}$ be an indexed open covering of X. An indexed family of continuous functions $\varphi_\alpha : X \to [0, 1]$ is a *partition of unity* on X dominated by $\{U_\alpha\}_{\alpha \in J}$ if:

(1) $\text{Supprt}(\varphi_\alpha) \subset U_\alpha$ for all $\alpha \in J$,

(2) the indexed family $\{\text{Support}(\varphi_\alpha)\}$ is locally finite, and

(3) $\sum_{\alpha \in J} \varphi_\alpha(x) = 1$ for each $x \in X$.

Note. We will show (in Theorem 41.7) that every paracompact Hausdorff space has a partition of unity $\{\varphi_\alpha\}_{\alpha \in J}$ for any given open covering. First, we need a lemma (which Munkres calls a "shrinking lemma" since, as we have seen before, a given cover is refined with "smaller" sets).

Lemma 41.6. Let X be a paracompact Hausdorff space. Let $\{U_\alpha\}_{\alpha \in J}$ be an indexed family of open sets covering X. Then there exists a locally finite indexed family $\{V_\alpha\}_{\alpha \in J}$ of open sets covering X such that $\overline{V}_\alpha \subset U_\alpha$ for all $\alpha \in J$.

Definition. For $\{U_\alpha\}_{\alpha \in J}$ an indexed family of open sets covering spcae X, a locally finite indexed family $\{V_\alpha\}_{\alpha \in J}$ of open sets covering X such that $\overline{V}_\alpha \subset U_\alpha$ for all $\alpha \in J$ is a *precise refinement* of $\{U_\alpha\}_{\alpha \in J}$.

Theorem 41.7. Let X be a paracompact Hausdorff space. Let $\{U_\alpha\}_{\alpha \in J}$ be an indexed open covering of X. Then there exists a partition of unity on X dominated by $\{U_\alpha\}_{\alpha \in J}$.

Theorem 41.8. Let X be a paracompact Hausdorff space. Let $\mathcal{C}$ be a collection of subsets of X and for each $X \in \mathcal{C}$ let $\varepsilon_C > 0$. If $\mathcal{C}$ is locally finite, then there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) > 0$ for all x, and $f(x) \leq \varepsilon_C$ for $x \in C$.

Revised: 10/30/2016