Chapter 7. Complete Metric Spaces and Function Spaces

Note. Recall from your Analysis 1 (MATH 4217/5217) class that the real numbers $\mathbb{R}$ are a "complete ordered field" (in fact, up to isomorphism there is only one such structure; see my online notes at `http://faculty.etsu.edu/gardnerr/4217/notes/1-3.pdf`). The Axiom of Completeness in this setting requires that every set of real numbers with an upper bound have a least upper bound. But this idea (which dates from the mid 19th century and the work of Richard Dedekind) depends on the ordering of $\mathbb{R}$ (as evidenced by the use of the terms "upper" and "least"). In a metric space, there is no such ordering and so the completeness idea (which is fundamental to all of analysis) must be dealt with in an alternate way. Munkres makes a nice comment on page 263 declaring that"completeness is a metric property rather than a topological one."

Section 43. Complete Metric Spaces

Note. In this section, we define Cauchy sequences and use them to define completeness. The motivation for these ideas comes from the fact that a sequence of real numbers is Cauchy if and only if it is convergent (see my online notes for Analysis 1 [MATH 4217/5217] `http://faculty.etsu.edu/gardnerr/4217/notes/2-3.pdf`; notice Exercise 2.3.13).

Definition. Let (X, d) be a metric space. A sequence (x_n) of points of X is a *Cauchy sequence* on (X, d) if for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that if $m, n \geq N$ then $d(x_n, x_m) < \varepsilon$. The metric space (X, d) is *complete* if every Cauchy sequence in X converges.

Note. By the Triangle Inequality for any metric, a convergent sequence is always Cauchy (whether the space is complete or not). If the space (X, d) is complete, then for A a closed subset of X, the subspace $(A, d|_A)$ is complete.

Note. It is not immediately clear how to address Cauchy sequences in a topological space. Convergence is easy since we have a limit to which to "anchor" open sets, but this is not the case when addressing Cauchy-ness. John von Neumann published "On Complete Topological Spaces" in 1935 (*Transactions of the American Mathematical Society* **37**(1), 1–20). This is available online at: `http://www.ams.org/journals/tran/1935-037-01/S0002-9947-1935-1501776-7/S0002-9947-1935-1501776-7.pdf` (accessed 10/10/2015). In this paper he addresses the idea of Cauchy sequences in metric spaces and comments: "The need of uniformity in [metric space] M arises from the fact that the elements of a fundamental sequence are postulated to be 'near to each other,' and not near to any fixed point. As a general topological space ... has no property which leads itself to the definition of such a 'uniformity,' it is impossible that a reasonable notion of 'completeness' could be defined in it." In this paper, von Neumann discusses total boundedness and comapctness is the setting of *topological linear spaces*. His definition of complete is then:

> Topological linear space L is *topologically complete* if every closed and totally bounded set $S \subset L$ is compact.

The 'uniformity' concern is dealt with by 'anchoring' open sets at the origin of the linear space (that is, using the zero vector 0): "However, linear spaces ..., even if only topological, afford a possibility of 'uniformization' for their topology: because of their homogeneity everything can be discussed in the neighborhood of 0."

Note. For metric space (X, d) define the standard bounded metric $\overline{d}(x, y) = \min\{d(x, y), 1\}$. Then a sequence (x_n) is Cauchy in (X, d) if and only if it is Cauchy in $(X, \overline{d})$ (since the Cauchy definition deals with "small" distances) and is convergent in (X, d) if and only if it is convergent in $(X, \overline{d})$. So space (X, d) is complete if and only if $(X, \overline{d})$ is complete.

Lemma 43.1. A metric space (X, d) is complete if every Cauchy sequence in X has a convergent subsequence.

Example 1. The metric space $\mathbb{Q}$ with the usual metric is *not* complete, since we can consider a sequence of rationals, (q_n), which converge to $\sqrt{2}$ (or any given irrational) in $\mathbb{R}$. Since the sequence converges in $\mathbb{R}$ then it is Cauchy in $\mathbb{R}$ and in $\mathbb{Q}$. But (q_n) does not converge in $\mathbb{Q}$. Think of a Cauchy sequence as a sequence which "wants" to to converge; in a complete space, it will converge. Cauchy sequences in non-complete spaces may not converge, informally, because the space has a "hole" at the point to which the sequence wants to converge!

Theorem 43.2. Euclidean space $\mathbb{R}^k$ (where $k \in \mathbb{N}$) is complete in either of its usual metrics, the Euclidean metric d or the square metric ρ.

Note. We saw in Section 20 that with the standard bounded metric on $\mathbb{R}$, $\overline{d}(a, b) = \min\{|a - b|, 1\}$, we have the following as a metric on $\mathbb{R}^\omega$:

$$D(\mathbf{x}, \mathbf{y}) = \sup_{i \in \mathbb{N}} \{\overline{d}(x_i, y_i)/i\}.$$

By Theorem 20.5, D induces the product topology on $\mathbb{R}^\omega$. We now show that $\mathbb{R}^\omega$ is complete relative to D. First we need a lemma about convergence of sequences in a product space.

Lemma 43.3. Let X be the product space $X = \prod_{\alpha \in J} X_\alpha$ (under the product topology) and let $(\mathbf{x}_n)$ be a sequence of points in X. Then $\mathbf{x}_n \to \mathbf{x}$ if and only if $\pi_\alpha(\mathbf{x}_n) \to \pi_\alpha(\mathbf{x})$ for all $\alpha \in J$.

Theorem 43.4. There is a metric for the product space $\mathbb{R}^\omega$ relative to which $\mathbb{R}^\omega$ is complete.

Note. $\mathbb{R}^\omega$ forms a *linear space* in the sense that for any $a, b \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\omega$ we have $a\mathbf{x} + b\mathbf{y} \in \mathbb{R}^\omega$ where we define $a\mathbf{x} + b\mathbf{y} = \mathbf{z}$ with $z_i = ax_i + by_i$ for all $i \in \mathbb{N}$. Since D is a metric on $\mathbb{R}^\omega$, we can define a *norm*, $\|\cdot\|$, on $\mathbb{R}^\omega$ as

$$\|\mathbf{x}\| = D(\mathbf{x}, \mathbf{0}) = \sup_{i \in \mathbb{N}} \{\overline{d}(x_i, 0)/i\} = \sup_{i \in \mathbb{N}} \{\min\{|x_i|, 1\}/i\} = \sup_{i \in \mathbf{N}} \{\min\{|x_i|/i, 1/i\}\}.$$

So $\mathbb{R}^\omega$ is a normed linear space (notice that $\|\mathbf{x}\| \leq 1$ for all $\mathbf{x} \in \mathbb{R}^\omega$). By Theorem

43.4, it is a complete normed linear space. A complete normed linear space is called a *Banach space*. So $\mathbb{R}^\omega$, along with $\|\cdot\|$, is an example of a Banach space. The "classical Banach spaces" are studied in our Real Analysis sequence (MATH 5210/5220) and based on Lebesgue integration theory. They are defined as

$$L^p(E) = \left\{ f \,\middle|\, \int_E |f|^p < \infty \right\} \text{ where } \|f\| = \left\{ \int_E |f|^p \right\}^{1/p}$$

for $1 \le p < \infty$. There is also an $L^\infty(E)$ space. The space $L^2(E)$ is also a complete inner product space (a complete inner product space is called a *Hilbert space*).

Note. In Example 21.1 it is shown that $\mathbb{R}^J$ where J is uncountable under the product topology is not metrizable. We now introduce a metric on $\mathbb{R}^J$ relative to which $\mathbb{R}^J$ is complete.

Definition. Let (Y, d) be a metric space. Let $\overline{d}(a, b) = \min\{d(a, b), 1\}$ be the standard bounded metric on Y derived from d. If $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha \in J}$ are points in Y^J then let

$$\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\overline{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}.$$

$\overline{\rho}$ is the *uniform metric* on Y^J corresponding to d.

Note. Recall that an element of Y^J is a function $f : J \to Y$, so that for $f, g \in Y^J$ we have the notation

$$\overline{\rho}(f, g) = \sup\{\overline{d}(f(\alpha), g(\alpha)) \mid \alpha \in J\}.$$

Theorem 43.5. If the space Y is complete in the metric d, then the space Y^J is complete in the uniform metric $\overline{\rho}$ corresponding to d.

Note. Since an element of Y^J is a function $f : X \to Y$, if X and Y are both topological spaces then $f \in Y^X$ is a function $f : X \to Y$ and we can test function f for continuity on space X. The set of all continuous functions from topological space X to topological space Y is denoted $\mathcal{C}(X, Y)$.

Definition. Let (Y, d) be a metric space and X a set. Function $f : X \to Y$ is *bounded* if $f(X)$ is a bounded subset of metric space (Y, d) (that is, there exists $M \in \mathbb{R}$ such that for all $y_1, y_2 \in f(X)$ we have $d(y_1, y_2) \leq M$). The set of all bounded functions from set X to metric space (Y, d) is denoted $\mathcal{B}(X, Y)$.

Note. The next result shows that if X is a topological space and Y is a complete metric space, then both $\mathcal{C}(X, Y)$ and $\mathcal{B}(X, Y)$ are complete in the uniform metric.

Theorem 43.6. Let X be a topological space and let (Y, d) be a metric space. The set $\mathcal{C}(X, Y)$ of continuous functions is closed in Y^X under the uniform metric. So is the set $\mathcal{B}(X, Y)$ of bounded functions. Therefore, if Y is a complete metric space, then both $\mathcal{C}(X, Y)$ and $\mathcal{B}(X, Y)$ are complete metric spaces under the uniform metric.

Note. If $E \subset \mathbb{R}$ then the set of all bounded functions mapping E into $\mathbb{R}$ forms a "linear space." A norm on this linear space if given by $\|f\|_\infty = \sup\{|f(x)| \mid x \in E\}$ and a metric is given by $d(f,g) = \|f-g\|_\infty = \sup\{|f(x)-g(x)| \mid x \in E\}$. You would encounter the is in the setting of "L^p spaces" in Real Analysis (MATH 5210/5220). We take this as motivation for the following definition.

Definition. Let (Y, d) be a metric space. For $f, g \in \mathcal{B}(X, Y)$ define

$$\rho(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\}.$$

This the the *sup metric* on $\mathcal{B}(X, Y)$.

Note. The sup metric is related to the uniform metric. Let $f, g \in \mathcal{B}(X, Y)$. If $\rho(f, g) > 1$ then there is some $x_0 \in X$ such that $d(f(x_0), g(x_0)) > 1$. Since $\overline{d}(f(x), g(x)) = \min\{d(f(x), g(x)), 1\}$ then $\overline{d}(f(x_0, g(x_0)) = 1$. Since $\overline{\rho}(f,g) = \sup\{\overline{d}(f(x), g(x)) \mid x \in X\}$ then, here, we $\overline{\rho}(f,g) = 1$. On the other hand, if $\rho(f,g) \le 1$ then $\overline{d}(f(x), g(x) = d(f(x), g(x)) \le 1$ and so $\overline{\rho}(f,g) = \rho(f,g)$. In either case, we have $\overline{\rho}(f,g) = \min\{\rho(f,g), 1\}$. This motivates the notation for $\overline{\rho}$, as first introduced in Section 20.

Note. The next result shows that every metric space can be embedded in a complete metric space. An alternative proof is outlined in Exercise 43.9.

Theorem 43.7. Let (X, d) be a metric space. There is an isometric embedding of X into a complete space.

Definition. Let X be a metric space. If $h : X \to Y$ is an isometric embedding of X into a complete metric space Y, then the subspace $\overline{h(X)}$ of Y is a complete metric space (a closed subspace of a complete space contains all of its limit points by the Sequence Lemma [Lemma 21.2], so every Cauchy sequence of elements of $\overline{k(X)}$ converges to an element of $\overline{h(X)}$) called the *completion* of X.

Note. In the previous definition, we use $\overline{h(X)}$ as the completion of X, as opposed to complete space Y, in order to make the completion the "smallest" complete space containing X (technically, containing the isometric image of X, $h(X)$). For example, the completion of $\mathbb{Q}$ is $\mathbb{R}$, although $\mathbb{R}^2$ is complete and contains $\mathbb{Q}$. In Exercise 43.10, it is shown that the completion of X is uniquely determined "up to an isometry."

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