Section 44. A Space-Filling Curve

Note. In this section, we give the surprising result that there is a continuous function from the interval $[0,1]$ onto the unit square $[0,1] \times [0,1]$.

Note. We first refer to Hans Sagan's *Space-Filling Curves* (Springer-Verlag, Universitext series, 1994) for some historical comments.

Note. In 1878, George Cantor proved that the interval $[0,1]$ and the square $[0,1] \times [0,1]$ have the same cardinality so that there is a one to one and onto function from $[0,1]$ to $[0,1] \times [0,1]$. In 1879, E. Netto proved that such a function must be discontinuous (in "Beitrag zur Mannigfaltigkeitslehre," *Crelle Journal*, **86**, 265–268 (1879)). In 1890, Guiseppe Peano (1858–1932) constructed a continuous onto mapping from $[0,1]$ to $[0,1] \times [0,1]$ ("Sur une courbe, qui remplit toute une aire plane," *Mathematische Annalen*, **36**(1), 157–160 (1890)). The result image of $[0,1]$ is called a "space-filling curve" (or "Peano curve") and satisfies the surprising property that the one-dimensional interval is continuously mapped onto the two-dimensional square! Additional examples were given by David Hilbert (1862–1943), Eliakim H. Moore (1862–1932), Henri Lebesgue (1875–1941), Wacław Sierpiński (1882–1969), George Pólya (1887–1985) and others [Sagan, page 1]. The references for these works are:

- David Hilbert, "Ueber die stetige Abbildung einer Line auf ein Flächenstück," *Mathematische Annalen*, **38**(3), 459–460 (1890).

- Eliakim H. Moore, "On certain crinkly curves," *Transactions of the Amer. Math. Society*, **1**, 72–90 (1900).
- Henri Lebesgue, *Le cons sur l'Intégration et la Recherche des Fonctions Primitives*, Gauthier-Villars, Paris, 44–45 (1904).
- Wacław Sierpiński, "Sur une nouvelle courve continue qui remplit toute une aire plane," *Bull. Acad. Sci. do Cracovie (Sci. math. et nat., Série A)*, 462–478 (1912).
- George Pólya, "Über eine Peanosche Kurve," *Bull. Acad. Sci. do Cracovie (Sci. math. et nat., Série A)*, 1–9 (1913).

Note. The exercises in this section have you show:

1. There is a continuous functions from $[0,1]$ onto $[0,1]^n$ for any given $n \in \mathbb{N}$ (Exercise 44.1).

2. There is a continuous function from $\mathbb{R}$ onto $\mathbb{R}^n$ for any give $n \in \mathbb{N}$ (Exercise 44.2).

3. There is not a continuous function from $\mathbb{R}$ onto $\mathbb{R}^\omega$ where $\mathbb{R}^\omega$ is given the product topology (Exercise 44.3).

Note. The Hahn-Mazurkiewicz Theorem states that: "A non-empty Hausdorff topological space is a continuous image of the unit interval if and only if it is a compact, connected, weakly locally connected, metrizable space." It is named for

Stefan Mazurkiewicz (1888–1945) and Hans Hahn (1879–1934). A Hausdorff space that is the continuous image of the closed unit interval is called a *Peano space*. See Exercise 44.4 for a little more information.

Theorem 44.1. Let $I = [0,1]$. There exists a continuous map $f : I \to I^2$ whose image fills up the entire square I^2 (that is, f is onto).

Proof. We follow Munkres' 4-step proof. In Steps 1 and 2 we define a sequence of continuous functions (f_n) where $f_n : I \to I^2$. In Step 3 we show that (f_n) is a Cauchy sequence and use the results of Section 43 to show that (f_n) converges to a continuous function $f : I \to I^2$. In Step 4 we show that f is onto.

Step 1. First, we map I into I^2 with $f_0 = g$ as shown in Figure 44.1. It is easy to express f_0 parametrically as

$$f_0(t) = \begin{cases} (t,t) \text{ for } t \in [0,1/2] \\ (t, 1-t) \text{ for } t \in (1/2, 1]. \end{cases}$$

We then modify $f_0 = g$ to produce $f_1 = g'$ as shown in Figure 44.2. Partition I into 8 pieces and partition I^2 into 4 pieces such that $f_1 = g'$ behaves similarly on the upper two squares and $f_1 = g'$ behaves on the lower two squares as shown. We could describe $f_1 = g'$ piecewise using 8 pieces.

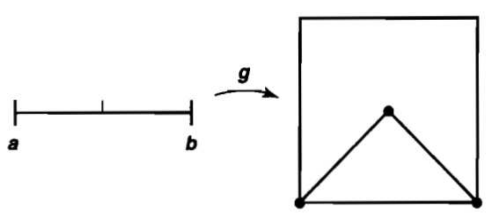

Figure 44.1

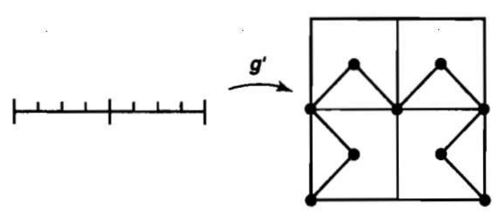

Figure 44.2

Step 2. We now partition each square of Figure 44.1 into 4 subsquares and produce f_2 as given in Figure 44.4. Each square is Figure 44.4 is further partitioned into 4 squares and f_3 produced as shown in Figure 44.5 (there are 64 subsquares in Figure 44.5, though only 16 of them are shown). The iterative process leads us to 4 cases in terms of how to define f_{n+1}, in terms of f. One of the cases is given in how g' is produced from g in Figures 44.1 and 44.2. The other three cases are symmetries of this case. If a square has f_n defined on it as given in Figure 44.1 rotated 180° (with the segment of f_n on this square starting and ending at the upper corners of the squares) then f_{n+1} on the subsquares is given in Figure 44.2 rotated 180°.

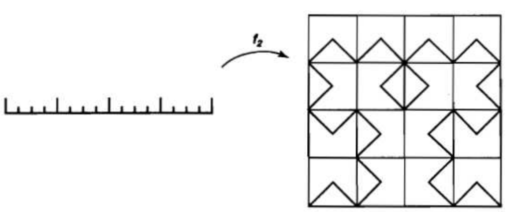

Figure 44.4

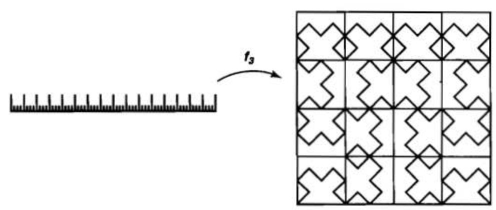

Figure 44.5

If a square has f_n defined as in Figure 44.3 then f_{n+1} is defined on the 4 subsquares as given by h' in Figure 44.3. If a square has f_n defined as it is given in Figure 44.3 rotated 180° (with the segment of f_n on the square starting and ending on the right corners of teh square) then f_{n+1} on the subsquares is given by Figure 44.3 rotated 180°. Notice that f_n is defined on 4^n subsquares, each containing 2 linear pieces of f_n (so f_n consists of 2×4^n linear pieces) and that the length of a side of each subsquare is $1/2^n$.

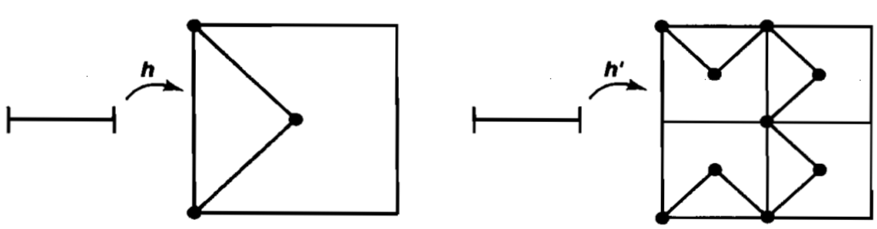

Figure 44.3

Step 3. Let $d(\mathbf{x},\mathbf{y})$ denote the square metric on $\mathbb{R}^2$:

$$d(\mathbf{x},\mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

(a metric on $\mathbb{R}^n$ introduced in Section 20). Let ρ denote the corresponding sup metric on (I, I^2):

$$\rho(f,g) = \sup\{d(f(t), g(t)) \mid t \in I\}.$$

By Theorem 43.2, $\mathbb{R}^2$ is complete under ρ. Since I^2 is closed in $\mathbb{R}^2$, then any Cauchy sequence in I^2 converges in $\mathbb{R}^2$, but the limit of the Cauchy sequence is a limit point of I^2 and since I^2 is closed, then the limit is in I^2. That is, any Cauchy sequence in I^2 converges in I^2. Hence I^2 is complete under ρ. By Theorem 43.6, $\mathcal{C}(I, I^2)$ is complete in the metric $\overline{\rho}$. Since a sequence is Cauchy under ρ if and only if it is Cauchy under $\overline{\rho}$ (see the note on page 264 or the "Note" in the class notes before Lemma 43.1), then $\mathcal{C}(I, I^2)$ is complete in the metric ρ.

We claim that (f_n) is defined piecewise on 4^n squares wach with a side of length $1/2^n$. Since f_{n+1} is based on f_n and each square is partitioned into 4 subsquares, then f_n under the square metric differed on each subsquare by at most $1/2^n$. So $\rho(f_n, f_{n+1}) \leq 1/2^n$. So by the Triangle Inequality

$$\rho(f_n, f_{n+m}) \leq \rho(f_n, f_{n+1}) + \rho(f_{n+1}, f_{n+2}) + \cdots + \rho(f_{n+m-1}, f_{n+m})$$

$$\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+m-1}} < \frac{1}{2^n} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+2}} + \cdots = \frac{1/2^n}{1 - 1/2} = \frac{2}{2^n} = \frac{1}{2^{n-1}}$$

for all $m, n \in \mathbb{N}$. So for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $1/2^{n-1} < \varepsilon$ and so for all $m, n \geq N$ we have $\rho(f_n, f_{n+m}) < \varepsilon$. That is, (f_n) is a Cauchy sequence with respect to ρ. Since $\mathcal{C}(I, I^2)$ is complete, then (f_n) converges in $\mathcal{C}(I, I^2)$ and so there is continuous $f : I \to I^2$ such that $(f_n) \to f$.

Step 4. Finally, we show that f is onto. Let $\mathbf{x} \in I^2$. For given $n \in \mathbb{N}$, $\mathbf{x}$ is in some subsquare with side of length $1/2^n$ and so there is a point $t_0 \in I$ such that $d(f(t_0), \mathbf{x}) \leq 1/2^n$. Let $\varepsilon > 0$. There there is $N \in \mathbb{N}$ such that $1/2^N < \varepsilon/2$ and so there is $t_0 \in I$ such that $d(f(t_0), \mathbf{x}) < \varepsilon$. Therefore $\mathbf{x}$ is a limit point of $f(I)$. Since f is continuous and I is compact (by Corollary 27.2) then $f(I)$ is compact by Theorem 26.5. By Theorem 26.3, $f(I)$ is therefore closed and so by Theorem 17.6 $f(I)$ includes its limit points. That is, $\mathbf{x} \in F(I)$ and so f is onto, as claimed.

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