Section 45. Compactness in Metric Spaces

Note. In this section we relate compactness to completeness through the idea of total boundedness (in Theorem 45.1). We define equicontinuity for a family of functions and use it to classify the compact subsets of $\mathcal{C}(X, \mathbb{R}^n)$ (in Theorem 45.4, the Classical Version of Ascoli's Theorem).

Note. Recall that, in a metric space, compactness, limit point compactness, and sequential compactness are equivalent (see Theorem 28.2). Lemma 43.1 states that a metric space in complete if every Cauchy sequence in X has a convergent subsequence. So if metric space X is sequentially compact (which is equivalent to compact) then, by definition, every sequence has a convergent subsequence and so by Lemma 43.1 metric space X is complete; that is, every compact metric space is complete. Of course, the converse does not hold (concisder $\mathbb{R}$). So we seek an additional condition on a complete space which will insure that it si compact. Inspired by the Heine-Borel Theorem (Theorem 27.3), we expect it to involve some kind of boundedness (note that a complete space is closed since it contains all of its limit points [by Theorem 17.6 and the definition of complete in terms of Cauchy sequences]).

Definition. A metric space (X, d) is *totally bounded* if for every $\varepsilon > 0$ there is a finite covering of X by ε-balls.

Example 45.1. Total boundedness implies boundedness for, with $\varepsilon = 1/2$, a covering of X with $B(x_1, 1/2), B(x_2, 1/2), \ldots, B(x_n, 1/2)$ shows that X has a diameter of at most $1 + \max\{d(x_i, x_j) \mid i, h \in \{1, 2, \ldots, n\}\}$. The converse does not hold as we see by considering $\mathbb{R}$ under the standard bounded metric $\overline{d}(a, b) = \min\{1, |a-b|\}$ (a covering with $\varepsilon = 1/2$ balls is the same with respect to $\overline{d}$ as it is with respect to $|\cdot|$ and there is not finite subcover); that is, $\mathbb{R}$ is bounded with respect fo $\overline{d}$ but not totally bounded.

Example 45.2. Under the metric $d(z, b) = |z - b|$, $\mathbb{R}$ is complete but not totally bounded, $(-1, 1)$ is totally bounded but not complete, and $[-1, 1]$ is both complete and totally bounded (and compact!).

Theorem 45.1. A metric space (X, d) is compact if and only if it is complete and totally bounded.

Note. We next explore compact subsets of $\mathcal{C}(X, \mathbb{R}^n)$ where we put the uniform topology on $\mathcal{C}(X, \mathbb{R}^n)$ (that is, the metric topology induced by the uniform metric $\overline{\rho}(f, g) = \sup\{\overline{d}(f(\mathbf{x}), g(\mathbf{x})) \mid \mathbf{x} \in \mathbb{R}^n\}$). We need an additional definition.

Definition. Let (Y, d) be a metric space. Let $\mathcal{F}$ be a subset of the function space $\mathcal{C}(X, Y)$. If $x_0 \in X$, the set function $\mathcal{F}$ of functions is *equicontinuous at* x_0 if given $\varepsilon > 0$ there is a neighborhood U of x_0 such that for all $x \in U$ and all $f \in \mathcal{F}$, $d(f(x), d(x_0)) < \varepsilon$. If the set $\mathcal{F}$ is equicontinuous at x_0 for each $x_0 \in X$ then set $\mathcal{F}$ is equicontinuous.

Lemma 45.2. Let X be a topological space and let (Y, d) be a metric space. If the subset $\mathcal{F}$ of $\mathcal{C}(X, Y)$ is totally bounded under the uniform metric corresponding to d, then $\mathcal{F}$ is equicontinuous under d.

Note. We need one more lemma before proving the classical version of Ascoli's Theorem.

Lemma 45.3. Let X be a topological space and let (Y, d) be a metric space. Assume X and Y are compact. If the subset $\mathcal{F}$ of $\mathcal{C}(X, Y)$ is equicontinuous under d, then $\mathcal{F}$ is totally bounded under the uniform and sup metrics corresponding to d.

Definition. If (Y, d) is a metric space, a subset $\mathcal{F}$ of $\mathcal{C}(X, Y)$ is *pointwise bounded* under d if for each $a \in X$, the subset $\mathcal{F}_a = \{f(a) \mid f \in \mathcal{F}\}$ of Y is bounded under d.

Theorem 45.4. The Classical Version of Ascoli's Theorem.
Let X be a compact space. Let $(\mathbb{R}^n, d)$ denote Euclidean space in either the square metric or the Euclidean metric. Give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding uniform topology. A subspace $\mathcal{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ has compact closure if and only if $\mathcal{F}$ is equicontinuous and pointwise bounded under d.

Corollary 45.5. Let X be compact. Let d denote either the square metric or the Euclidean metric on $\mathbb{R}^n$. Give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding uniform topology. A subspace $\mathcal{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ is compact if and only if it is closed, bounded under the sup metric ρ, and equicontinuous under d.

Note. Arzela's Theorem, given in the exercises, shifts the focus from a subspace of $\mathcal{C}(X, \mathbb{R}^n)$ to a sequence in $\mathcal{C}(X, \mathbb{R}^n)$.

Exercise 45.3. Arzela's Theorem. Let X be compact. Let $f_n \in \mathcal{C}(X, \mathbb{R}^k)$. If the collection $\{f_n\}$ is pointwise bounded and equicontinuous, then the sequence f_n has a uniformly convergent subsequence (and so the limit of the subsequence is continuous)

Note. In section 47, we generalize the Classical Version of Ascoli's Theorem as follows.

Theorem 47.1. Ascoli's Theorem. Let X be a topological space and let (Y, d) be a metric space. Give $\mathcal{C}(X, Y)$ the topology of compact convergence. let $\mathcal{F}$ be a subset of $\mathcal{C}(X, Y)$.

(a) If $\mathcal{F}$ is equicontinuous under d and the set $\mathcal{F}_a = \{f(a) \mid f \in \mathcal{F}\}$ has compact closure for each $a \in X$, then $\mathcal{F}$ is contained in a compact subspace of $\mathcal{C}(X, Y)$.

(b) The converse holds if X is locally compact Hausdorff.

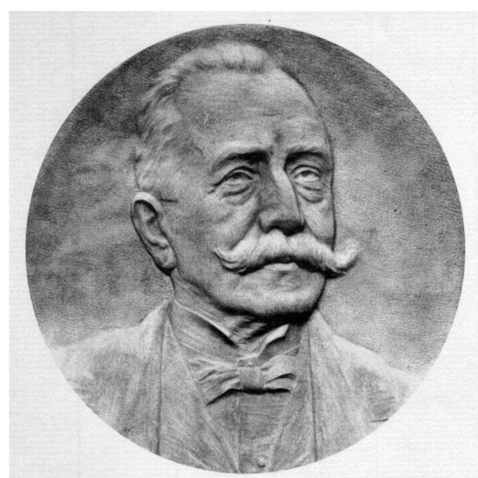
Note. This in turn leads to a generalization of Arzela's Theorem.

Exercise 47.4. Arzela's Theorem, General Version. Let X be a Hausdorff space that is σ-compact. Let f_n be a sequence of functions $f_n : X \to \mathbb{R}^k$. If the collection $\{f_n\}$ is pointwise bounded and equicontinuous, then the sequence f_n has a subsequence that converges, in the topology of compact convergence, to a continuous function.

Note. Giulio Ascoli, a 19th century Italian mathematician, introduced the idea of equicontinuity in 1884. In 1889 another Italian mathematician, Cesare Arzelà, generalized Ascoli's Theorem into the ArzelàAscoli Theorem concerning convergent subsequences of equicontinuous sequences of continuous functions. These comments are based on the Wikipedia pages for "The Arzelà-Ascoli Theorem" and "Giulio Ascoli."

Giulio Ascoli

January 20, 1843–July 12, 1896



Cesare Arzelà

March 6, 1847–March 15, 1912

Note. In Royden and Fitzpatrick's *Real Analysis*, 4th edition (Pearson/Prentice Hall, 2010), the text used in our Real Analysis sequence (MATH 5210-5220), the following version of Ascoli's and Arzelà's work is given (in Section 10.1):

The Arzelà-Ascoli Theorem. Let X be a compact metric space and $\{f_n\}$ is uniformly bounded, equicontinuous sequence of real-valued functions on X. Then $\{f_n\}$ has a subsequence that converges uniformly on X to a continuous function f on X.

Note. In Conway's *Functions of One Complex Variable I*, 2nd edition (Springer-Verlag, 1978), the text use in our Complex Analysis sequence (MATH 5510-5520), continuous functions mapping an open set in $\mathbb{C}$ to a metric space (Ω, d) is addressed. S subset $\mathcal{F} \subset \mathcal{C}(G, \Omega)$ is defined to be *normal* if each sequence in $\mathcal{F}$ has a subsequence which converges to a function f in $\mathcal{C}(G, \Omega)$. We then have (in Section VII.1):

The Arzelà-Ascoli Theorem. A set $\mathcal{F} \subset \mathcal{C}(G, \Omega)$ is normal if and only if the following two conditions are satisfied:

(a) For each $z \in G$, we have that $\{f(z) \mid f \in \mathcal{F}\}$ has compact closure in ω;

(b) $\mathcal{F}$ is equicontinuous at each point of G.

Note. In Promislow's *A First Course in Functional Analysis* (John Wiley & Sons, 2008), the text we use in our Fundamentals of Functional Analysis (MATH 5740) class, a set if defined to be *relatively compact* if its closure is compact. We then have (in Section 9.2):

The Arzelà-Ascoli Theorem. If S is a compact metric space, a subset A of $\mathcal{C}(S)$ (the set of continuous real valued or complex valued functionals on S) is relatively compact if and only if it is bounded and equicontinuous.

Revised: 7/1/2017