Section 45. Compactness in Metric Spaces

Note. In this section we relate compactness to completeness through the idea of total boundedness (in Theorem 45.1). We define equicontinuity for a family of functions and use it to classify the compact subsets of $C(X, \mathbb{R}^n)$ (in Theorem 45.4, the Classical Version of Ascoli’s Theorem).

Note. Recall that, in a metric space, compactness, limit point compactness, and sequential compactness are equivalent (see Theorem 28.2). Lemma 43.1 states that a metric space in complete if every Cauchy sequence in $X$ has a convergent subsequence. So if metric space $X$ is sequentially compact (which is equivalent to compact) then, by definition, every sequence has a convergent subsequence and so by Lemma 43.1 metric space $X$ is complete; that is, every compact metric space is complete. Of course, the converse does not hold (consider $\mathbb{R}$). So we seek an additional condition on a complete space which will insure that it is compact. Inspired by the Heine-Borel Theorem (Theorem 27.3), we expect it to involve some kind of boundedness (note that a complete space is closed since it contains all of its limit points [by Theorem 17.6 and the definition of complete in terms of Cauchy sequences]).

Definition. A metric space $(X, d)$ is totally bounded if for every $\varepsilon > 0$ there is a finite covering of $X$ by $\varepsilon$-balls.
Example 45.1. Total boundedness implies boundedness for, with $\varepsilon = 1/2$, a covering of $X$ with $B(x_1, 1/2), B(x_2, 1/2), \ldots, B(x_n, 1/2)$ shows that $X$ has a diameter of at most $1 + \max\{d(x_i, x_j) \mid i, h \in \{1, 2, \ldots, n\}\}$. The converse does not hold as we see by considering $\mathbb{R}$ under the standard bounded metric $d(a, b) = \min\{1, |a-b|\}$ (a covering with $\varepsilon = 1/2$ balls is the same with respect to $d\overline{d}$ as it is with respect to $| \cdot |$ and there is not finite subcover); that is, $\mathbb{R}$ is bounded with respect to $d\overline{d}$ but not totally bounded.

Example 45.2. Under the metric $d(z, b) = |z-b|$, $\mathbb{R}$ is complete but not totally bounded, $(-1, 1)$ is totally bounded but not complete, and $[-1, 1]$ is both complete and totally bounded (and compact!).

Theorem 45.1. A metric space $(X, d)$ is compact if and only if it is complete and totally bounded.

Note. We next explore compact subsets of $\mathcal{C}(X, \mathbb{R}^n)$ where we put the uniform topology on $\mathcal{C}(X, \mathbb{R}^n)$ (that is, the metric topology induced by the uniform metric $\rho(f, g) = \sup\{d(f(x), g(x)) \mid x \in \mathbb{R}^n\}$). We need an additional definition.

Definition. Let $(Y, d)$ be a metric space. Let $\mathcal{F}$ be a subset of the function space $\mathcal{C}(X, Y)$. If $x_0 \in X$, the set function $\mathcal{F}$ of functions is equicontinuous at $x_0$ if given $\varepsilon > 0$ there is a neighborhood $U$ of $x_0$ such that for all $x \in U$ and all $f \in \mathcal{F}$, $d(f(x), d(x_0)) < \varepsilon$. If the set $\mathcal{F}$ is equicontinuous at $x_0$ for each $x_0 \in X$ then set $\mathcal{F}$ is equicontinuous.
Lemma 45.2. Let $X$ be a topological space and let $(Y, d)$ be a metric space. If the subset $\mathcal{F}$ of $\mathcal{C}(X, Y)$ is totally bounded under the uniform metric corresponding to $d$, then $\mathcal{F}$ is equicontinuous under $d$.

Note. We need one more lemma before proving the classical version of Ascoli’s Theorem.

Lemma 45.3. Let $X$ be a topological space and let $(Y, d)$ be a metric space. Assume $X$ and $Y$ are compact. If the subset $\mathcal{F}$ of $\mathcal{C}(X, Y)$ is equicontinuous under $d$, then $\mathcal{F}$ is totally bounded under the uniform and sup metrics corresponding to $d$.

Definition. If $(Y, d)$ is a metric space, a subset $\mathcal{F}$ of $\mathcal{C}(X, Y)$ is pointwise bounded under $d$ if for each $a \in X$, the subset $\mathcal{F}_a = \{f(a) \mid f \in \mathcal{F}\}$ of $Y$ is bounded under $d$.

Theorem 45.4. The Classical Version of Ascoli’s Theorem.
Let $X$ be a compact space. Let $(\mathbb{R}^n, d)$ denote Euclidean space in either the square metric or the Euclidean metric. Give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding uniform topology. A subspace $\mathcal{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ has compact closure if and only if $\mathcal{F}$ is equicontinuous and pointwise bounded under $d$. 
Corollary 45.5. Let $X$ be compact. Let $d$ denote either the square metric or the Euclidean metric on $\mathbb{R}^n$. Give $C(X, \mathbb{R}^n)$ the corresponding uniform topology. A subspace $\mathcal{F}$ of $C(X, \mathbb{R}^n)$ is compact if and only if it is closed, bounded under the sup metric $\rho$, and equicontinuous under $d$.

**Note.** Arzela’s Theorem, given in the exercises, shifts the focus from a subspace of $C(X, \mathbb{R}^n)$ to a sequence in $C(X, \mathbb{R}^n)$.

**Exercise 45.3. Arzela’s Theorem.** Let $X$ be compact. Let $f_n \in C(X, \mathbb{R}^k)$. If the collection $\{f_n\}$ is pointwise bounded and equicontinuous, then the sequence $f_n$ has a uniformly convergent subsequence (and so the limit of the subsequence is continuous).

**Note.** In section 47, we generalize the Classical Version of Ascoli’s Theorem as follows.

**Theorem 47.1. Ascoli’s Theorem.** Let $X$ be a topological space and let $(Y, d)$ be a metric space. Give $C(X, Y)$ the topology of compact convergence. let $\mathcal{F}$ be a subset of $C(X, Y)$.

(a) If $\mathcal{F}$ is equicontinuous under $d$ and the set $\mathcal{F}_a = \{f(a) \mid f \in \mathcal{F}\}$ has compact closure for each $a \in X$, then $\mathcal{F}$ is contained in a compact subspace of $C(X, Y)$.

(b) The converse holds if $X$ is locally compact Hausdorff.
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Note. This in turn leads to a generalization of Arzela’s Theorem.

Exercise 47.4. Arzela’s Theorem, General Version. Let $X$ be a Hausdorff space that is $\sigma$-compact. Let $f_n$ be a sequence of functions $f_n : X \to \mathbb{R}^k$. If the collection $\{f_n\}$ is pointwise bounded and equicontinuous, then the sequence $f_n$ has a subsequence that converges, in the topology of compact convergence, to a continuous function.

Note. Giulio Ascoli, a 19th century Italian mathematician, introduced the idea of equicontinuity in 1884. In 1889 another Italian mathematician, Cesare Arzelà, generalized Ascoli’s Theorem into the Arzelà-Ascoli Theorem concerning convergent subsequences of equicontinuous sequences of continuous functions. These comments are based on the Wikipedia pages for “The Arzelà-Ascoli Theorem” and “Giulio Ascoli.”
Note. In Royden and Fitzpatrick’s *Real Analysis*, 4th edition (Pearson/Prentice Hall, 2010), the text used in our Real Analysis sequence (MATH 5210-5220), the following version of Ascoli’s and Arzelà’s work is given (in Section 10.1):

**The Arzelà-Ascoli Theorem.** Let $X$ be a compact metric space and $\{f_n\}$ is uniformly bounded, equicontinuous sequence of real-valued functions on $X$. Then $\{f_n\}$ has a subsequence that converges uniformly on $X$ to a continuous function $f$ on $X$.

Note. In Conway’s *Functions of One Complex Variable I*, 2nd edition (Springer-Verlag, 1978), the text used in our Complex Analysis sequence (MATH 5510-5520), continuous functions mapping an open set in $\mathbb{C}$ to a metric space $(\Omega, d)$ is addressed. A subset $\mathcal{F} \subset C(G, \Omega)$ is defined to be normal if each sequence in $\mathcal{F}$ has a subsequence which converges to a function $f$ in $C(G, \Omega)$. We then have (in Section VII.1):

**The Arzelà-Ascoli Theorem.** A set $\mathcal{F} \subset C(G, \Omega)$ is normal if and only if the following two conditions are satisfied:

(a) For each $z \in G$, we have that $\{f(z) \mid f \in \mathcal{F}\}$ has compact closure in $\omega$;

(b) $\mathcal{F}$ is equicontinuous at each point of $G$. 
Note. In Promislow’s *A First Course in Functional Analysis* (John Wiley & Sons, 2008), the text we use in our Fundamentals of Functional Analysis (MATH 5740) class, a set if defined to be *relatively compact* if its closure is compact. We then have (in Section 9.2):

**The Arzelà-Ascoli Theorem.** If $S$ is a compact metric space, a subset $A$ of $\mathcal{C}(S)$ (the set of continuous real valued or complex valued functionals on $S$) is relatively compact if and only if it is bounded and equicontinuous.