Munkres Chapter 9.
The Fundamental Group

Note. These supplemental notes are based on James R. Munkres' *Topology*, 2nd edition, Prentice Hall (2000).

Note. We are interested in when two topological spaces are homeomorphic. There is no general method to determine when there is such a homeomorphism. However, if we can find a property which homeomorphic spaces share, and show that this property is *not* shared be two spaces, then we know the spaces are not homeomorphic. For example, $[0,1]$ and $(0,1)$ (with the usual subspace topology inherited from $\mathbb{R}$) are no homeomorphic because $[0,1]$ is compact and $(0,1)$ is not. In this chapter we associate a group (called the fundamental group) with a topological space. We then can show that certain spaces are not homeomorphic because they have different fundamental groups.

Section 51. Homotopy of Paths

Note. When we define the fundamental group of a topological space (in Section 52) the elements of the group will be closed paths (technically, equivalence classes of closed paths). In this section, we define *path* and what it means for two paths to be equivalent in a topological space.

Note. From now on in these supplemental notes, we drop the specific topology when refering to a topological space. That is, we refer to "space X" instead of "topological space $(X, \mathcal{T})$."

Definition. If f and f' are continuous maps of the space X into the space Y, we say that f is *homotopic* to f' if there is a continuous map $F : X \times I \to Y$ such that

$$F(x, 0) = f(x) \text{ and } F(x, 1) = f'(x)$$

for all $x \in X$, where $I = [0, 1]$. (Do not confuse f' with the derivative of f—it is simply some continuous function.) The map F is a homotopy between f and f'. If f is homotopic to f', we will write $f \simeq f'$. If $f \simeq f'$ and f' is a contant map, then f is said to be *nulhomotopic*.

Note. We think of homotopy $F(x, t)$ as continuously deforming f to f', starting at f when $t = 0$ and finishing at f' when $t = 1$.

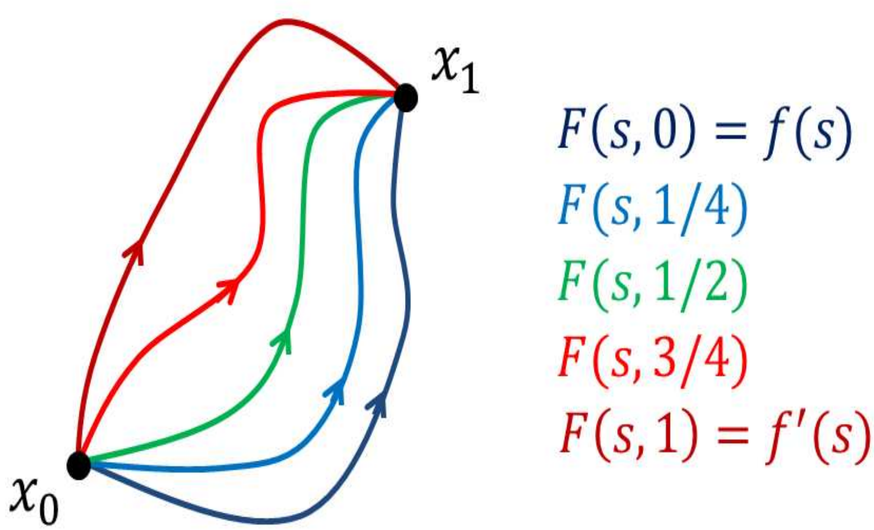
Definition. If $f : [0, 1] \to X$ is continuous and $f(0) = x_0$, $f(1) = x_1$, then f is a *path from x_0 to x_1*.

Definition. Two paths f and f' mapping the interval $I = [0, 1]$ into space X are *path homotopic* if they have the same initial point $x_0 = f(0) = f'(0)$ and the same final point $x_1 = f(1) = f'(1)$, and if there is a continuous map $F : I \times I \to X$ such that

$$F(s, 0) = f(s) \quad \text{and} \quad F(s, 1) = f'(s)$$
$$F(0, t) = x_0 \quad \text{and} \quad F(1, t) = x_1$$

for each $s \in I$ and each $t \in I$. F is then called a *path homotopy* between f and f'. If f is path homotopic to f' then we denote this as $f \simeq_p f'$.

Note. As with homotopy of continuous maps, path homotopy is a continuous deformation of path f to path f':



Note. The idea of path homotopy plays a big role in complex analysis. If two paths in the complex plane are path homotopic (and of bounded variation—such paths are called *rectifiable*), then the complex integral of an analytic function over one path equals the complex integral of the analytic function over the other path. This is called "Cauchy's Theorem." For more details, see the notes for Complex Analysis 1 and 2 (MATH 5510/5520): `http://faculty.etsu.edu/gardnerr/5510/notes/IV-6.pdf`.

Lemma 51.1. The relations $\simeq$ and $\simeq_p$ are equivalence relations.

Note. Since $\simeq_p$ is an equivalence relation, then the equivalence classes of $\simeq_p$ partition the set of paths. We denote the equivalence class containing path f as $[f]$.

Example 51.1. Let f and g be any two maps of a space X into $\mathbb{R}^2$. Then

$$F(x,t) = (1-t)f(x) + tg(x)$$

is a homotopy between f and g. It is called a *straight line homotopy* since each point on the graph of $f(x)$ is mapped along a straight line as t varies from 0 to 1 to a corresponding point on the graph of $g(x)$. If A is a *convex* subspace of $\mathbb{R}^n$ (that is, for any two points in set A the line segment joining the points is also in set A) then any two paths from x_0 to x_1 are path homotopic under the straight line homotopy.

Example 51.2. Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$ be the *punctured plane.* Consider the paths in X from $(1,0)$ to $(-1,0)$:

$$f(s) = (\cos(\pi s), \sin(\pi s)),$$

$$g(s) = (\cos(\pi s), 2\sin(\pi s)),$$

$$h(s) = (\cos(\pi s), -\sin(\pi s)).$$

Then f and g are path homotopic in X—in fact the straight line homotopy holds between f and g. However, the straight line homotopy does not hold between f and h (since this would require us to map the point $(0,1)$ on f to the point $(0,-1)$ on h by passing through $(0,0)$). This alone does not show that f and h are not path homotopic, but they are *not*! We'll return to this example later.

Note. In defining the fundamental group of a space, we need a binary operation on the equivalence classes of paths. In that direction, we have the following.

Definition. If f is a path in X from x_0 to x_1 and g is a path in X from x_1 to x_2, we define the *product* $f * g$ of f and g to be the path h from x_0 to x_2 given by

$$h(s) = \begin{cases} f(2s) & \text{for } s \in [0,1/2] \\ g(2s-1) & \text{for } s \in (1/2,1]. \end{cases}$$

Lemma 51.A. The operation $*$ on equivalence classes of paths defined as $[f]*[g] = [f*g]$ is well-defined. That is, if $f \in [f]$, $g \in [g]$, and $f*g \in [f*g]$, then for any $f' \in [f]$ and $g' \in [g]$, we have $f' * g' \in [f*g]$.

Proof. Let $f, f' \in [f]$ and $g, g' \in [g]$. Say F is a path homotopy between f and f' and G is a path homotopy between g and g'. Define

$$H(s) = \begin{cases} F(2s, t) & \text{for } s \in [0, 1/2] \\ G(2s-1, t) & \text{for } s \in (1/2, 1]. \end{cases}$$

Then H is a path homotopy between $f * g$ and $f' * g'$. ∎

Note. The product of two paths is only defined, say $f * g$, when $f(1) = x_1 = g(0)$. So we cannot make a group our of all equivalence classes of paths since the binary operation between equivalence classes is sometimes not defined. However, when the binary operation is defined then we get the usual type of group behavior (associativity, identities, and inverses). An algebraic structure satisfying the properties given in the next result is called a *groupoid*.

Theorem 51.2. The operation $*$ on the equivalence classes of paths in space X satisfies the following properties:

(1) Associativity: If $[f] * ([g] * [h])$ is defined, then so is $([f] * [g]) * [h]$ and they are equal.

(2) Right and left Identities: Given $x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all of I to the point x. If f is a path in X from x_0 to x_1 then $[f] * [e_{x_1}] = [f]$ and $[e_{x_0}] * [f] = [f]$.

(3) Inverses: Given the path f in X from x_0 to x_1 let $\overline{f}$ be the path defined by $\overline{f}(s) = f(1-s)$. Then $\overline{f}$ is called the *reverse* of f, $[f] * [\overline{f}] = [e_{x_0}]$ and $[\overline{f}] * [f] = [e_{x_1}]$.

Note. We can inductively extend the proof for associativity to a product of more than three paths to get the following.

Theorem 51.3. Let f be a path in X, and let $a_0, a_1, \ldots, a_n$ be numbers such that $0 = a_0 < a_1 < \cdots < a_n = 1$. Let $f_i : I \to X$ be the path that equals the positive linear map of I onto $[a_{i-1}, a_i]$ followed by f. Then

$$[f] = [f_1] * [f_2] * \cdots * [f_n].$$

Revised: 1/11/2018