Section 52. The Fundamental Group

Note. In the previous section, we defined a binary operation on *some* paths in a topological space. That is, $f * g$ is defined only if $f(1) = g(0)$. We modify things in this section by considering paths that all have the same initial and final point, say x_0 (called a "base point"). We then use the binary operation of the previous section on the equivalence classes of these types of paths to define the fundamental group. We show that homeomorphic topological spaces have isomorphic fundamental groups.

Note. Munkres reviews a few ideas from abstract algebra on pages 330 and 331, including homomorphism, kernel, coset, normal subgroup, and quotient group.

Note. Recall that Theorem 51.2 shows that the set of all paths in a topological space forms a groupoid (a "groupoid" because the product of some pairs of paths are not defined). If we restrict our attention to paths that all have the same point as their initial and final point, then the product will always be defined and Theorem 51.2 implies that the result will be a group.

Definition. Let X be a space and $x_0 \in X$. A path in X that begins and ends at x_0 is a *loop* based at x_0. The set of path homotopy equivalence classes of loops based at x_0, with the operation $*$ of Section 51, is the *fundamental group* of X relative to the *base point* x_0. It is denoted $\pi_1(X, x_0)$.

Note. Sometimes the fundamental group is called the *first homotopy group* of X. There is a homotopy group $\pi_n(X, x_0)$ for each $n \in \mathbb{N}$ and this is part of the area called homotopy theory.

Example 52.1. If f is a loop in $\mathbb{R}^n$ based at x_0, then there is a straight-line homotopy between f and the constant path at x_+0. So $\pi_1(\mathbb{R}^n, x_0)$ is the trivial group $\{e\}$. Similarly, if X is any convex subset of $\mathbb{R}^n$, then $\pi_1(X, x_0)$ is the trivial group.

Note. We now want to show that the fundamental group does not depend on the base point.

Definition. Let α be a path in X from x_0 to x_1. Define $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ as

$$\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha].$$

Note. The picture of $\hat{\alpha}$ is (from page 332 of Munkres):

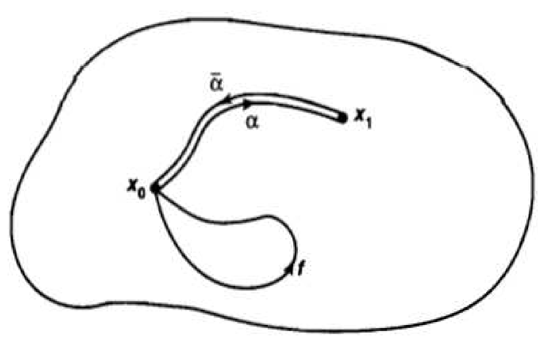

Figure 52.1

Theorem 52.1. The map $\hat{\alpha}$ is a group isomorphism.

Corollary 52.2. If X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

Note. If a space is not path connected, then for x_0 in a component of X, $\pi_1(X, x_0)$ provides no information about the other component of X. So the study of fundamental groups is usually restricted to path connected spaces.

Note. If a space is path connected then all fundamental groups $\pi_1(X, x_i)$ are isomorphic for any $x_i \in X$. But the isomorphism between the groups depends on the base points. So we do not speak of "*the* fundamental group" without reference to a base point. In fact, the isomorphism of $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$ is independent of the path if and only if these fundamental groups are abelian (see Exercise 52.3).

Note. The following definition of "simply connected" takes advantage of the little bit of algebraic topology which we have developed.

Definition. A space X is *simply connected* if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial group for some $x_0 \in X$ (and hence, by Corollary 52.2, for all $x_0 \in X$). This case is often denoted $\pi_1(X, x_0) = 0$.

Note. The following result deals with general "paths," as opposed to "loops."

Lemma 52.3. In a simply connected space X, any two paths having the same initial and final points are path homotopic.

Note. We now develop the equipment to show that the fundamental group is a topological invariant. That is, it is a property shared by homeomorphic spaces. Of course, this has to be done in terms of base points.

Definition. Let $h : X \to Y$ be a continuous map between spaces X and Y and let $y_0 = h(x_0)$. Then for f a loop in X based at x_0, $h \circ f : I \to Y$ is a loop in Y based at y_0. We denote this as $h : (X, x_0) \to (Y, y_0)$. Define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $h_*[f] = [h \circ f]$. Then h_* is the *homomorphism induced by* h relative to the base point x_0.

Note. Map h_* is well defined, since for $f, f' \in [f]$, there is a path homotopy F between f and f'. Then $h \circ F$ is a path homotopy between $h \circ f$ and $h \circ f'$. That is, $h \circ f \simeq_p h \circ f'$ and $h \circ f, h \circ f' \in [h \circ f]$. By equation $(*)$ in the proof of Theorem 51.2, we have that $(h \circ f) * (h \circ g) = h \circ (f * g)$ from which $(h \circ [f]) * (h \circ [g]) = h \circ ([f] * [g])$ by the definition of $*$ on the equivalence classes. So h is, in fact, a group homomorphism.

Note. In the event that we consider the homomorphism induced by h relative to different base points, we denote h_* as $(h_{x_0})_*$ or $(h_{x_1})_*$, etc.

Theorem 52.4. If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

Note. Munkres refers to the property given in Theorem 52.4 and the property given in the next result as the *functorial properties* of the induced homomorphism. But the next result is the **big result** concerning fundamental groups and their applications.

Corollary 52.5. If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X and Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.

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