Section 53. Covering Spaces

Note. The covering space introduced in this section will be useful in computing some fundamental groups that are nontrivial.

Definition. Let $p : E \to B$ be a continuous onto map. The open set U of B is *evenly covered* by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets $V_\alpha \subseteq E$ such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ is a partition of $p^{-1}(U)$ into *slices.*

Note. Munkres describes $p^{-1}(U)$ as a "stack of pancakes" each having the same size and shape as U:

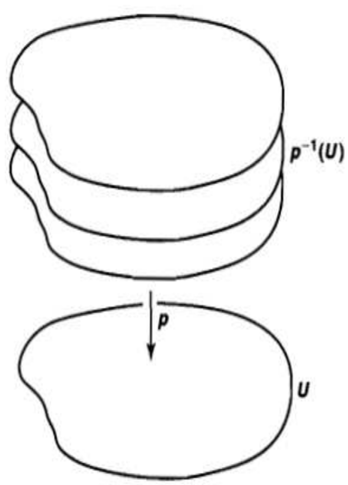


Figure 53.1

We will see below that an example of an even covering of the unit circle results by wrapping the real line around the circle, in which case an open set on the circle has an inverse image consisting of a bunch of open intervals in $\mathbb{R}$ (these open intervals are the "pancakes").

Definition. Let $p : E \to B$ be continuous and onto. If every point $b \in B$ has a neighborhood U that is evenly covered by p, then p is a *covering map* and E is said to be a *covering space* of B.

Lemma 53.A. Let $p : E \to B$ be a covering map. Then p is an open map (that is, p maps open sets to open sets).

Example 53.1. Let X be a space and define $E = X \times \{1, 2, \ldots, n\}$ consisting of n disjoint copies of X. The map $p : E \to X$ given by $p(x, i) = x$ for all i is a covering map and E is a covering space of X.

Theorem 53.1. The map $p : \mathbb{R} \to S^1$ (the "1-sphere") given by the equation $p(x) = (\cos(2\pi x), \sin(2\pi x))$ is a covering map.

Note. If $p : E \to B$ is a covering map, then p is a *local homeomorphism* of E with B. That is, for each $e \in E$ there is a neighborhood of e that is mapped homeomorphically by p onto an open subset of B.

Example 53.2. The map $p : \mathbb{R}^+ \to S^1$ given by the equation

$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

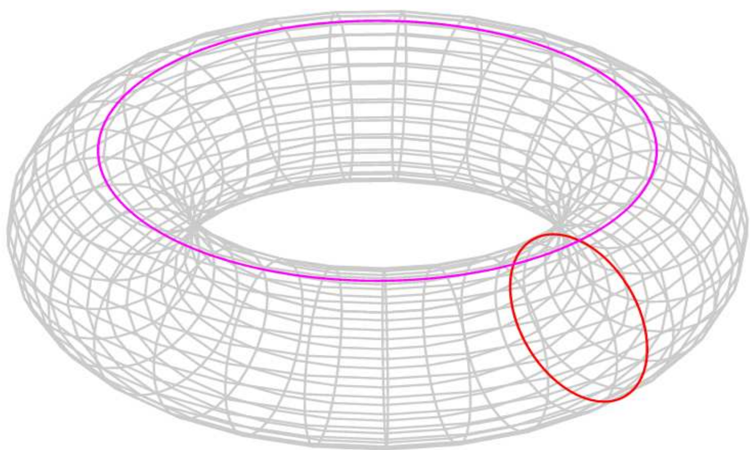
is onto and is a local homeomorphism. But p is not a covering map because of the behavior of the point $b_0 = (1, 0)$. A small ε-neighborhood of b_0 has an inverse image of small neighborhoods V_n for each $n \in \mathbb{N}$ which are mapped homeomorphically onto the neighborhood of b_0, but $V_0 = (0, \delta)$ is not; $p(V_0)$ does not even contain b_0. So p is a local homeomorphism, but not a covering map (because of its failure to evenly cover an ε-neighborhood of b_0). This example also shows that the restriction of a covering map (the map of Theorem 53.1 on $\mathbb{R}$ restricted to $\mathbb{R}^+$ here) may not be a covering map. The following result gives a condition under which a restriction of a covering map is a covering map.

Theorem 53.2. Let $p : E \to B$ be a covering map. If B_0 is a subspace of B, and if $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \to B_0$ obtained by restricting p to E_0 is a covering map of B_0.

Theorem 53.3. If $p : E \to B$ and $p' : E' \to B'$ are covering maps, then $p \times p' : E \times E' \to B \times B'$ is a covering map.

Example 53.4. The space $T = S^1 \times S^1$ is the *torus*. The product map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ is a covering map of T by $\mathbb{R}^2$ by Theorem 53.5 where p is the covering map of S^1 by $\mathbb{R}$ as given in Theorem 53.1. We typically think of S^1 as a subset of $\mathbb{R}^2$, so this representation of the torus "lives" in $\mathbb{R}^4$.

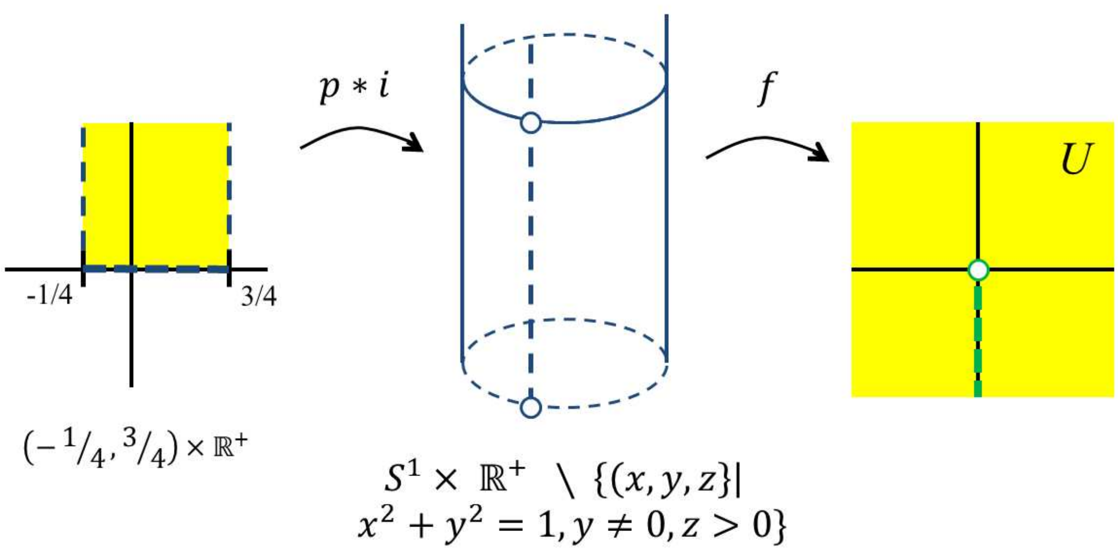
An alternate representation of the torus is as the image of $\mathbb{R}^2$ under the covering map $f(x, y) = ((R + r\cos(2\pi x))\cos(2\pi y), (R + r\cos x)\sin(2\pi y), r\sin x)$, where $R > r$. This gives the torus embedded in $\mathbb{R}^3$ as a circle of radius r rotated around a circle of radius R.



From `http://en.wikipedia.org/wiki/Torus`, accessed 12/8/2014.

Also see Munkres Figure 53.5. The representation of the torus (which Munkres calls "the familiar doughnut-shaped surface D) is a subspace of $\mathbb{R}^3$. In fact, $S^1 \times S^1$ is homeomorphic to D. We will see in Section 60 that the fundamental group of the torus is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Example 53.6. Consider the covering map $p * i : \mathbb{R} \times \mathbb{R}^+ \to S^1 \times \mathbb{R}^+$ where i is the identity map of $\mathbb{R}^+$ and p is the map of $\mathbb{R}$ onto S^1 of Theorem 53.1. Define $f : S^1 \times \mathbb{R}^+ \to \mathbb{R} \setminus \{(0,0)\}$ as $f(\vec{x}, t) = t\vec{x}$ (where we represent an element of S^1 as a 2-dimensional real vector). Then f is a homeomorphism—it takes the infinite half-cylinder $S^1 \times \mathbb{R}^+$ and "peels" it open to produce the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$. Now if we compose $p * i$ with f we get a homeomorphism between the open upper half plane $\mathbb{R} \times \mathbb{R}^+$ with the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$. In fact, this is a covering map. For open $U \subseteq \mathbb{R}^2 \setminus \{(0,0)\}$ with $U = \{(x,y) \in \mathbb{R}^2 \mid \text{ when } x = 0 \text{ then } y > 0\}$, the slices are of the form $(-(1/4)n, (3/4)n) \times \mathbb{R}^+$:



This covering map appears in Complex Analysis (MATH 5510/5520) in relation to the Riemann surface associated with the complex logarithm function.

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