Section 54. The Fundamental Group of the Circle

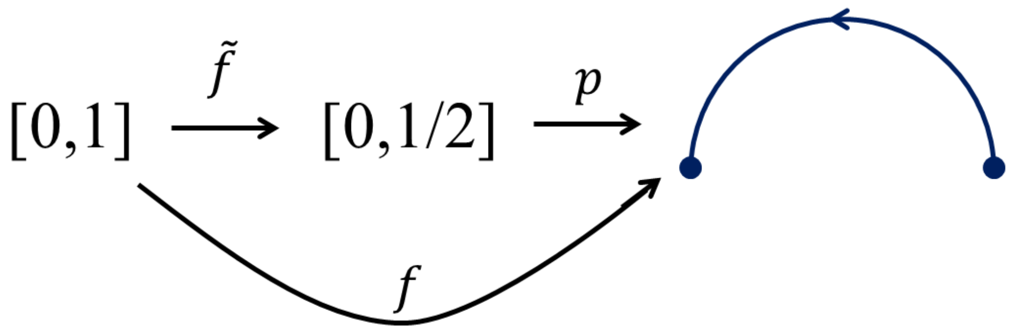
Note. In this section, we tie together the idea of a covering space from the previous section with the idea of the fundamental group from Section 52. We also show that the fundamental group of the circle S^1 is isomorphic to $\mathbb{Z}$.

Definition. Let $p : E \to B$ be a map between space E and space B. If f is a continuous mapping of some space X into B, then a *lifting* of f is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

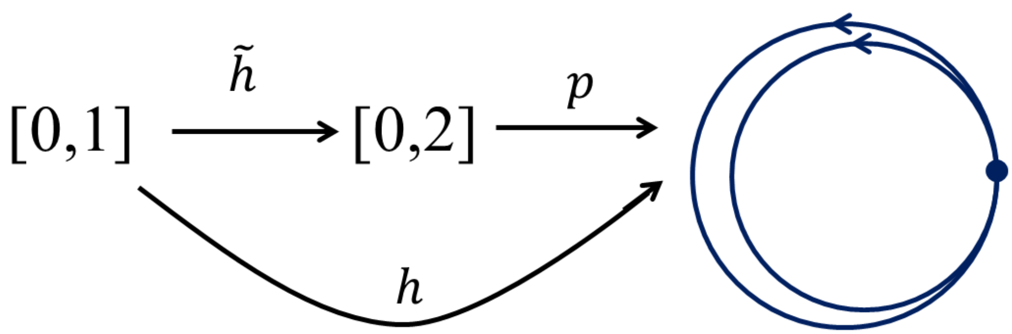
Note. The diagram for these maps is:

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[dr, "f"'] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

Example 54.1. Consider the covering $p : \mathbb{R} \to S^1$ of Theorem 53.1. The path $f : [0,1] \to S^1$ given by $f(s) = (\cos(\pi s), \sin(\pi s))$ (a path along S^1 from $b_0 = (1,0)$ to $(-1,0)$) *lifts* to the path $\tilde{f}(s) = s/2$ in $\mathbb{R}$ beginning at 0 and ending at $1/2$, because $p \circ \tilde{f}$ maps $[0,1]$ to the upper half of S^1 and f maps $[0,1]$ to this same upper half of S_1:



The path $g(s) = (\cos(\pi s), -\sin(\pi s))$ lifts to the path $\tilde{g}(s) = -s/2$ beginning at 0 and ending at $-1/2$ (as above, but now involving the lower half of S^1). The path $h(s) = (\cos(4\pi s), \sin(4\pi s))$ lifts to the path $\tilde{h}(s) = 2s$ beginning at 0 and ending at 2:



Note. In the next two results, we show that for a covering space, (1) paths can be lifted, and (2) path homotopies can be lifted.

Lemma 54.1. Let $p : E \to B$ be a covering map, and let $p(e_0) = b_0$. Any path $f : [0,1] \to B$ beginning at b_0 has a unique lifting to a path f in E beginning at e_0.

Lemma 54.2. Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. Let the map $F : I \times I \to B$ be continuous with $F(0,0) = b_0$. There is a unique lifting of F to a continuous map $\tilde{F} : I \times I \to E$ such that $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

Note. The next result shows that homotopic paths are lifted to homotopic paths.

Theorem 54.3. Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1. Let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths in E beginning at e_0. If f and g are path homotopic, then $\tilde{f}$ and $\tilde{g}$ end at the same point of E and are path homotopic.

Note. We define a mapping which will be useful in determining the fundamental group of the circle S^1.

Definition. Let $p : E \to B$ be a covering map. Let $b_0 \in B$. Choose e_0 so that $p(e_0) = b_0$. Given an element $[f]$ of $\pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path in E that begins at e_0 (we know by Lemma 54.1 that $\tilde{f}$ is well-defined in terms of any $f \in [f]$). Let $\varphi([f])$ denote the end point $\tilde{f}(1)$ of $\tilde{f}$. Then

$$\varphi : \pi_1(B, b_0) \to p^{-1}(b_0).$$

is the *lifting correspondence* derived from the covering map p.

Theorem 54.4. Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence

$$\varphi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

is surjective (onto). If E is simply connected, it is bijective.

Theorem 54.5. The fundamental group of S^1 is isomorphic to the additive group of integers, $\mathbb{Z}$.

Note. The previous result justifies our intuitive idea that the fundamental group of S^1 is generated by starting at $(1, 0)$ and creating loops that wrap around S^1 a positive integer number of times (counterclockwise) and loops that wrap around S^1 a negative integer number of times (clockwise). The intuitive pasting together of loops corresponds to the binary operation in the fundamental group.

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