Section 55. Retractions and Fixed Points

Note. We now use the fact that the fundamental group of S^1 in $\mathbb{Z}$ to prove several results concerning the unit circle S^1 and the unit ball (or disk) B^2, including the Brouwer Fixed Point Theorem for the Disk (Theorem 55.6).

Definition. If $A \subseteq X$, then a *retraction* of X onto A is a continuous map $r : X \to A$ such that $r|_A$ is the identity map of A. If such a map r exists, we say that A is a *retract* of X.

Example. A retract of the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$ to the circle $S^1 = \{(x,y) \mid x^2+y^2=1\}$ (technically, the right had side is the inclusion map which maps S^1 into $\mathbb{R}^2$) is given by

$$r((x,y)) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right).$$

Another retraction can be found by omitting the square roots in the denominator in r.

Lemma 55.1. If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion $j : A \to X$ is injective (one to one).

Theorem 55.2. No-Retraction Theorem.

There is no retraction of the closed disk B^2 to the circle S^1.

Lemma 55.3. Let $h : S^1 \to X$ be a continuous map. Then the following are equivalent.

(1) h is nulhomotopic.

(2) h extends to a continuous map $k : B^2 \to X$.

(3) The homomorphism h_* induced by h is the trivial homomorphism of fundamental groups.

Corollary 55.4. The inclusion map $j : S^1 \to \mathbb{R} \setminus \{(0,0)\}$ is not nulhomotopic. The identity map $i : S^1 \to S^1$ is not nulhomotopic.

Note. We now introduce a new idea (that of "vector field") in a special setting (the unit disk). This idea will be used to prove the Brouwer Fixed Point Theorem for the Disk.

Definition. A *vector field* on B^2 is an ordered pair $(\vec{x}, v(\vec{x}))$ where $\vec{x} \in B^2$ and v is a continuous map of B^2 into $\mathbb{R}$. A vector field is *nonvanishing* if $v(\vec{x}) \neq 0$ for all $\vec{x} \in B^2$.

Theorem 55.5. Given a nonvanishing vector field on B^2, there exists a point of S^1 where the vector field points directly inward and a point of S^1 where the vector field points directly outward.

Note. The proof of the following result is quiet easy, given the theory we have developed.

Theorem 55.6. The Brouwer Fixed-Point Theorem for the Disk. If $f : B^2 \to B^2$ is continuous, then there exists a point $\vec{x} \in B^2$ such that $f(\vec{x}) = \vec{x}$.

Proof. ASSUME to the contrary that $f(\vec{x}) \neq \vec{x}$ for every $\vec{x} \in B^2$. Then define $v(\vec{x}) = f(\vec{x}) - \vec{x}$. So $(\vec{x}, v(\vec{x}))$ is a nonvanishing vector field in B^2. Then the vector field must point directly outward at some point $\vec{x}$ of S^1 by Theorem 55.5, we say $v(\vec{x}) = a\vec{x}$ where $a > 0$. Then $v(\vec{x}) = f(\vec{x}) - \vec{x} = a\vec{x}$ or $f(\vec{x}) = (1 + a)\vec{x}$. But then $f(\vec{x}) \notin B^2$, a contradiction. This contradiction shows that the assumption is false, and so $f(\vec{x}) = \vec{x}$ for some $\vec{x} \in B^2$. ∎

Note. The Brouwer Fixed Point Theorem for the Disk is often popularly described as follows. Take two circular pieces of paper and place one directly on top of the other. Now pick up the top piece and wad it up. Place it back on top of the original piece. Then there will be at least one point on the wadded paper which lies directly over the point where it was initially. In fact, the paper need not be circular, but could be rectangular.

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