Section 56. The Fundamental Theorem of Algebra

Note. In our graduate program, you have several opportunities to see a proof of the Fundamental Theorem of Algebra:

(1) It is easily proven in Complex Variables [MATH 4337/5337] and Complex Analysis [MATH 5510/5520] using Liouville's Theorem (in fact, the Fraleigh text which we use in Introduction to Modern Algebra [MATH 4127/5127, 4137/5137] fives this proof). It can also be proved using Rouche's Theorem.

(2) It is proven almost entirely algebraically (two results from analysis must be "borrowed") in Modern Algebra 2 [MATH 5420].

(3) Another opportunity is available now to us here!

The proof given here is based on the fact that the fundamental group of S^1 is $\mathbb{Z}$. So the this proof is based on algebraic topology.

Theorem 56.1. The Fundamental Theorem of Algebra.
A polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0$$

of degree $n \geq 1$ with real or complex coefficients has at least one (real or complex) root.

Proof. We break the proof into four steps.
Step 1. Consider the map $f : S^1 \to S^1$ given by $f(z) = z^n$ where we interpret S^1 as the unit circle in $\mathbb{C}$, $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Let $p_0 : I \to S^1$ be the "standard loop" in S^1:

$$p_0(s) = e^{2\pi i s} = \cos(2\pi s) + i\sin(2\pi s) = (\cos(2\pi s), \sin(2\pi s)) \text{ where } s \in [0,1]$$

and we interpret elements of $\mathbb{C}$ as elements of $\mathbb{R}^2$; of course these are homeomorphic topological spaces. The image of the standard loop under f is the loop

$$f(p_0(s)) = (e^{2\pi i s})^n = \cos(2\pi n s) + i\sin(2\pi n s) = (\cos(2\pi n s), \sin(2\pi n s)) \text{ for } s \in [0,1].$$

This loop lifts to the path $\ell(s) = ns$, $s \in [0,1]$, in the covering space $\mathbb{R}$, because

$$p_0 \circ \ell(s) = p_0(ns) = e^{2\pi i(ns)} = (\cos(2\pi n s), \sin(2\pi n s)) \text{ where } s \in [0,1].$$

With φ as the lifting correspondence of Theorem 54.4 and Theorem 54.5, we have

$$\varphi : \pi_1(S^1, b_0) \to p^{-1}(b_0) = \mathbb{Z} \text{ where } b_0 = (1,0).$$

For path g, $\varphi([g])$ is the endpoint of the lifting of g ($\tilde{g}(1)$ in the notation of Section 54), so the loop $f \circ p_0$ corresponds through φ to the integer n. The loop p_0 corresponds to the integer 1.

Now consider the homomorphism f_* induced by the continuous function f. Since $f : S^1 \to S^1$ then $f_* : \pi_1(S^1, b_0) \to \pi_1(S^1, b_0)$. Now, the equivalence class $[p_0]$ is a generator of the domain group of f_*, $\pi_1(S^1, b_0)$. So the behavior of f_* on $[p_0]$ determines the behavior of f_* on $\pi_1(S^1, b_0)$. Now $f_*([p_0]) = [f \circ p_0]$ by definition (see page 333 of Munkres or Section 52 page 4 of these class notes). As commented above, $f \circ p_0$ corresponds through the lifting correspondence φ to $n \in \mathbb{Z}$. Munkres concludes that f_* is "multiplication by n." More precisely, if for $[g], [h] \in \pi_1(S^1, b_0)$ we have $f_*([g]) = [h]$, then $\varphi([h]) = n\varphi([g])$, or $\varphi(f_*([g])) = n\varphi([g])$.

Suppose $[g_1] \neq [g_2]$. Since φ is one to one as shown in the proof of Theorem 54.5, then $\varphi([g_1]) \neq \varphi([g_2])$, and so we have

$$\varphi(f_*([g_1])) = n\varphi([g_1]) \neq n\varphi([g_2]) = \varphi(f_*([g_2]))$$

(notice that $n \neq 0$ since we hypothesized $n \geq 1$). Since φ is a function and $\varphi(f_*([g_1])) \neq \varphi(f_*[g_2]))$, then $f_*([g_1]) \neq f_*([g_2])$. We have shown that $[g_1] \neq [g_2]$ implies $f_*([g_1]) \neq f_*([g_2])$. Therefore $f_* : \pi_1(S^1, b_0) \to \pi_1(S^1, b_0)$ is one to one (injective).

Step 2. Let $g : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ be the map $g(z) = z^n$. Then map g equals the map f of Step 1 followed by the inclusion map $j : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$. Now f_* is one to one by Step 1. Let j_* be the homomorphism induced by j. Since S^1 is a retract of $\mathbb{R}^2 \setminus \{(0,0)\}$, then j_* is one to one by Lemma 55.1. Therefore $g_* = (j \circ f)_* = j_* \circ f_*$ by Theorem 52.4, and so g_* is one to one and $g_* : \pi_1(S^1, b_0) \to \pi_1(\mathbb{R}^2 \setminus \{(0,0)\}, a_0)$. Since $\pi_1(S^1, b_0) \cong \mathbb{Z}$ by Theorem 54.5, then g_* cannot be the trivial homomorphism. So, by Theorem 55.3 (the contrapositive of "(1) implies (3)") g is not nulhomotopic.

Step 3. We now prove (by contradiction) a special case of the Fundamental Theorem. For the given polynomial equation, suppose

$$|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0| < 1. \qquad (*)$$

ASSUME the original polynomial equation has no root in the closed unit disk B^2 in $\mathbb{C}$. Then we can define a map $k : B^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$ by the equation

$$k(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

Of course, k is continuous on B^2. Let h be the restriction of k to S^1. Then h extends to a continuous map on B^2 (namely, map k), so by Lemma 55.3, h is nulhomotopic.

On the other hand, define $F : S^1 \times I \to \mathbb{R}^2 \setminus \{(0,0)\}$ by the equation

$$F(z,t) = z^n + t(a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0).$$

Then $F(z,0) = z^n = g(z)$ for $z \in S^1$, and $F(z,1) = z_n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = h(z)$ for $z \in S^1$. Now $g(z)$ is never $(0,0)$ since $g(z) \in S^1$ for all $z \in S^1$. Also $h(z)$ is never $(0,0)$ since its extension $k(z)$ to B^2 is never $(0,0)$ by the assumption. Now for $0 < t < 1$ we have

$$\begin{aligned}
|F(z,t)| &= |z^n + t(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)| \\
&\geq |z^n| - |t(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)| \text{ by the Triangle Inequality in } \mathbb{C} \\
&\geq |z^n| - t(|a_{n-1}z^{n-1}| + \cdots + |a_1 z| + |a_0|) \text{ by the Triangle Inequality} \\
&= 1 - t(|a_{n-1}| + \cdots + |a_1| + |a_0|) \text{ since } z \in S^1 \text{ and so } |z| = 1 \\
&> 0 \text{ by } (*).
\end{aligned}$$

So $F(z,t)$ is a homotopy from g to h and $g \simeq_p h$. Since h is nulhomotopic (i.e., homotopic to a constant path) then g is nulhomotopic. But in Step 2, we saw that g *is* nulhomotopic, so we have a contradiction. This contradiction shows the assumption that the polynomial equation has no zero in B^2 is false. Hence the polynomial equation under the restriction $(*)$ has a zero in B^2.

Step 4. We now prove the Fundamental Theorem for a general polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0.$$

Choose real number $c > 0$ sufficiently large so that

$$\left|\frac{a_{n-1}}{c}\right| + \left|\frac{a_{n-2}}{c^2}\right| + \cdots + \left|\frac{a_1}{c^{n-1}}\right| + \left|\frac{a_0}{c^n}\right| < 1$$

(this can be done since the limit as $c \to \infty$ of each of the summands is 0). Let $x = cy$. The polynomial equation then becomes

$$(cy)^n + a_{n-1}(cy)^{n-1} + \cdots + a_2(cy)^2 + a_1(cy) + a_0 = 0$$

or (dividing both sides by c^n)

$$y^n + \frac{a_{n-1}}{c}y^{n-1} + \cdots + \frac{a_2}{c^{n-2}}y^2 + \frac{a_1}{c^{n-1}}y + \frac{a_0}{c^n} = 0.$$

Now the polynomial on the left hand side satisfies condition $(*)$ by the choice of c. So by Step 3, this equation has a root, say $y = y_0$. Then $x_0 = cy_0$ is a root of the original general polynomial equation. ∎

Note. As stated here, the Fundamental Theorem of Algebra implies that every complex polynomial (that is, every element of the polynomial ring $\mathbb{C}[x]$) has a complex zero. This property is sometimes called "algebraically closed." So the Fundamental Theorem of Algebra can be restated as "The complex number field $\mathbb{C}$ is algebraically closed."

Note. By the Factor Theorem (see, for example, John B. Fraleigh's *A First Course in Abstract Algebra*, 7th edition, Corollary 23.3), $x = a$ is a zero of polynomial $p(x)$ if and only if $(x - a)$ is a factor of $p(x)$. It follows by induction that any polynomial of degree n with complex coefficients can be factored into a product of n linear complex terms (counting multiplicity). This is also sometimes taken as a statement of the Fundamental Theorem of Algebra.

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