Section 57. The Borsuk-Ulam Theorem

Note. Munkres introduces this section by stating a "brain teaser." I prefer motivation based on Exercise 57.1: "At any given moment in time, there exists a pair of antipodal points on the surface of the earth at which both the temperature and the barometric pressure are equal." This is a consequence of the fact that both temperature and barometric pressure (and windspeed, humidity, etc.) are continuous functions from S^2 into $\mathbb{R}$.

Definition. If $\vec{x}$ is a point of S^n, then its *antipode* is the point $-\vec{x}$. A map $h : S^n \to S^m$ is *antipode-preserving* if $h(-\vec{x}) = -h(\vec{x})$ for all $\vec{x} \in S^n$. Notice that we are interpreting S^n as embedded in $\mathbb{R}^{n+1}$:

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}.$$

Theorem 57.1. If $h : S^1 \to S^1$ is continuous and antipode-preserving, then h is not nulhomotopic.

Theorem 57.2. There is not continuous antipode-preserving map $g : S^2 \to S^1$.

Theorem 57.3. Borsuk-Ulam Theorem for S^2

Given a continuous map $f : S^2 \to \mathbb{R}^2$, there is a point $\vec{x}$ of S^2 such that $f(\vec{x}) = f(-\vec{x})$.

Note. We can now justify the claim made at the beginning of this section. With S^2 as the surface of the Earth and the continuous function f that associates an ordered pair consisting of temperature and barometric pressure, the Borsuk-Ulam Theorem implies that there are two antipodal points on the surface of the Earth with the values of both temperature and barometric pressure equal. The Borsuk-Ulam Theorem can be generalized to n-dimensions: Any continuous function from an n-sphere, S^n, to $\mathbb{R}^n$ must send some pair of antipodal points of S^n to the same point of $\mathbb{R}^n$.

Theorem 57.4. The Bisection Theorem.
Given two bounded polygonal regions in $\mathbb{R}^2$, there exists a line in $\mathbb{R}^2$ that bisects each of them.

Note. Theorem 57.4 can be generalized to: The volumes of any n measurable n-dimensional solids can always be simultaneously bisected by a $(n-1)$-dimensional hyperplane. This is often called the Stone-Tukey Theorem since a proof for $n > 3$ was given by A.H. Stone and J.W. Tukey in 1942 ("Generalized Sandwich Theorems," *Duke Mathematics Journal*, **9**, 356–359). When $n = 3$, this is commonly called the "Ham Sandwich Theorem" since it implies that, given two pieces of bread and a piece of ham (i.e., measurable subsets of $\mathbb{R}^3$), there is a way to cut the resulting sandwich with one swipe of the knife (i.e., a plane) that bisects the two pieces of bread and the ham. See `mathworld.wolfram.com/HamSandwichTheorem.html`.

Note. One of my colleagues has addressed the Ham Sandwich Theorem in the following:

“The Spirit is Willing but the Ham is Rotten”

by John Kinloch and Rick Norwood.

The American Mathematical Monthly, 101 (5), May 1994, Page 470

The Spirit is Willing but the Ham is Rotten

John Kinloch and Rick Norwood

What is wrong with the following “proof” of the Ham Sandwich Theorem?

Theorem: Given any sandwich composed of bread, ham, and cheese, there is a single plane which cuts the sandwich into two parts, such that the two parts contain equal amounts of bread, equal amounts of ham, and equal amounts of cheese.

Proof: Consider the bread. It has a center of mass. Call this point p. Let q be the center of mass of the ham. Let r be the center of mass of the cheese. There is a plane containing p, q, and r. This plane divides the sandwich into two parts containing equal amounts of bread, equal amounts of ham, and equal amounts of cheese.

The theorem is true. The “proof” is fallacious. Why?

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