Section 58. Deformation Retracts and Homotopy Type

Note. In Section 54, we used covering spaces and covering maps of Section 53 to show that the fundamental group of S^1 is isomorphic to $\mathbb{Z}$. In this section, we introduce homotopy type and use it to compute the fundamental group of one space in terms of the fundamental group of another space. In particular, we show that the fundamental group of S^n is isomorphic to the fundamental group of $\mathbb{R}^{n+1} \setminus \{\vec{0}\}$ (in Theorem 58.2).

Lemma 58.1. Let $h, k : (X, x_0) \to (Y, y_0)$ be continuous maps. If h and k are homotopic and if the image of the base point $x_0 \in X$ remains fixed at y_0 during the homotopy then the induced homomorphisms h_* and k_* are equal.

Note. In Section 54, we saw that the fundamental group S^1 is isomorphic to $\mathbb{Z}$ (Theorem 54.5). In the next section we show that the fundamental group of S^n for $n \geq 2$ is trivial (Theorem 59.3). These results allow us to find the fundamental group of some other spaces using the following result.

Theorem 58.2. The inclusion map $j : S^n \to \mathbb{R}^{n+1} \setminus \{\vec{0}\}$ induces an isomorphism of fundamental groups $\pi_1(S^n, x_0)$ and $\pi_1(\mathbb{R}^{n+1} \setminus \{\vec{0}\}, x_0)$, for $n \geq 1$.

Definition. Let A be a subspace of X. Then A is a *deformation retract* of X if the identity map of X is homotopic to a map that carries all of X into A, such that each point of A remains fixed during the homotopy. That is, there is a continuous $H : X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a$ for all $a \in A$ and $t \in I$. The homotopy H is a *deformation retraction* of X onto A.

Note. A deformation retract differs from a retraction in that the deformation retract requires the fixing of the elements of A throughout the mapping.

Note. If the homotopy H is a deformation retraction of X onto A, then the map $r : X \to Z$ defined as $r(x) = H(x, 1)$ is a retraction of X onto A. Also, as in the proof of Theorem 58.2, H is a homotopy between the identity map of X (given by $H(x, 0) = x$) and the map $j \circ r$ where $j : A \to X$ is inclusion (given by $H(x, 1) = r(x) \in A$).

Note. The proof of Theorem 58.2 carries over to give the following.

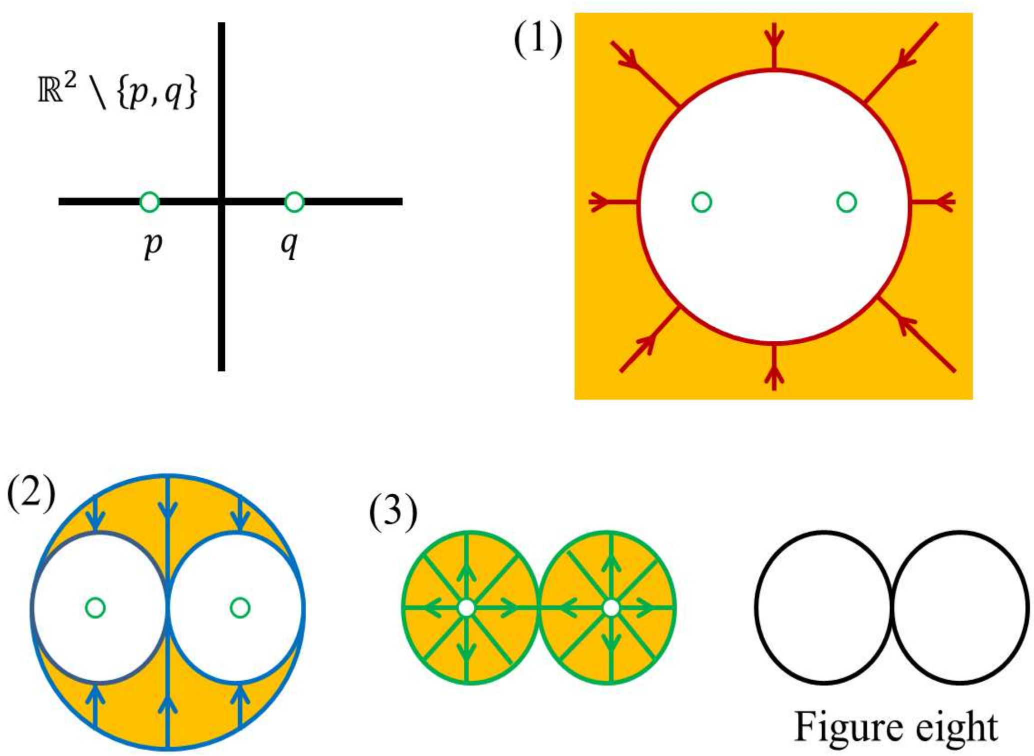
Theorem 58.3. Let A be a deformation retract of X. Let $x_0 \in A$. Then the inclusion map $j : (A, x_0) \to (X, x_0)$ induces an isomorphism between $\pi_1(A, x_0)$ and $\pi_1(X, x_0)$.

Example 1. Let B denote the z-axis in $\mathbb{R}^3$. Consider the space $\mathbb{R}^3 \setminus B$. The map

$$H(x, y, z, t) = (x, y, (1-t)z)$$

is a deformation retract of $\mathbb{R}^3 \setminus B$ to $(\mathbb{R}^2 \setminus \{\vec{0}\}) \times \{0\}$. By Theorem 58.3, we know that the fundamental group of $\mathbb{R}^3 \setminus B$ is isomorphic to the fundamental group of $\mathbb{R}^2 \setminus \{\vec{0}\}$. By Theorem 58.2, we know that the fundamental group of $\mathbb{R}^2 \setminus \{\vec{0}\}$ is isomorphic to the fundamental group of S^1. Theorem 54.5 gives that the fundamental group of S^1 is isomorphic to $\mathbb{Z}$. So the fundamental group of $\mathbb{R}^3 \setminus B$ is isomorphic to $\mathbb{Z}$.

Example 2. Consider $\mathbb{R}^2 \setminus \{p, q\}$ $(p \neq q)$, the *doubly punctuated plane*. It has a "figure eight" space as a deformation retract as indicated in the following "three stage" deformation:

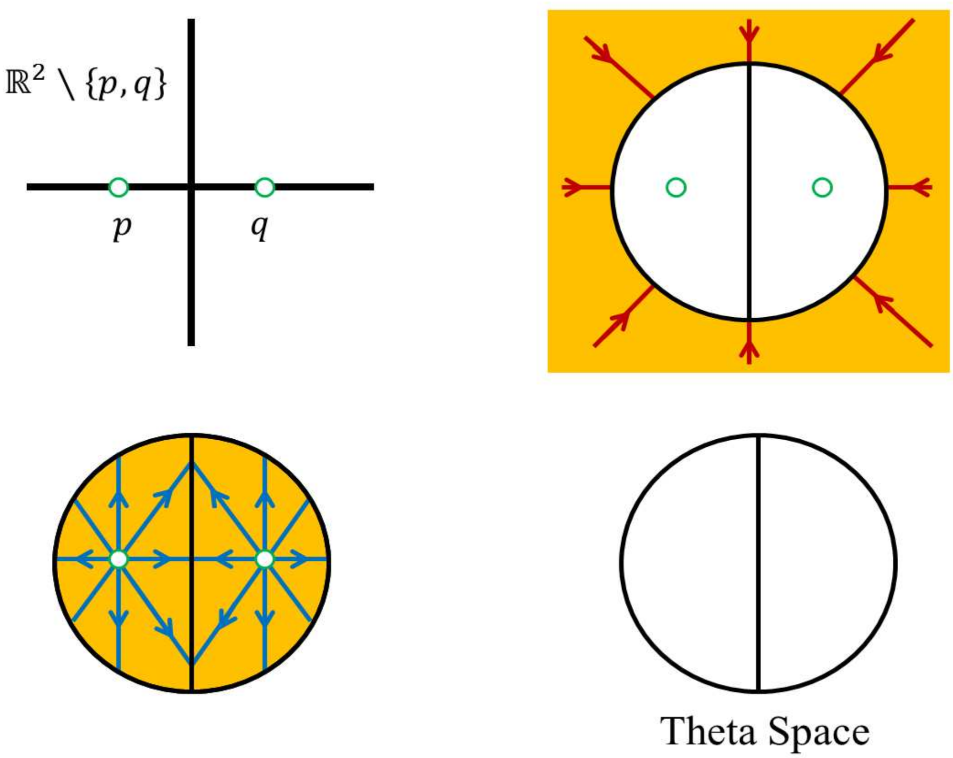


So by Theorem 58.3, the fundamental group of the doubly punctuated plane is isomorphic to the fundamental group of the figure eight. In Section 60 we argue that the fundamental group of the figure eight is a free group on two generators (i.e., the free product of two infinite cyclic groups—see page 5 of the class notes for Section 60).

Example 3. Another deformation of the doubly punctuated plane is the "theta space"

$$\theta = S^1 \cup (\{0\} \times [-1, 1]).$$

The deformation retraction is given as follows:



Again, Theorem 58.3 tells us that the theta space and the doubly punctuated plane have isomorphic fundamental groups.

Note. The figure eight and theta space have isomorphic fundamental groups. However, neither space is a deformation retract of the other (consider the "bar" in the theta space and the point in the center of the figure eight space).

Definition. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps. Suppose that the map $g \circ f : X \to X$ is homotopic to the identity map of X and the map $f \circ g : Y \to Y$ is homotopic to the identity map of Y. Then the maps f and g are *homotopy equivalences* and each is the *homotopy inverse* of the other.

Note. If $f : X \to Y$ and $h : Y \to Z$ are homotopy equivalences then $h \circ f : X \to Z$ is a homotopy equivalence. So the relation of homotopy equivalence is an equivalence relation.

Definition. If spaces X and Y are homotopy equivalent then they are of the same *homotopy type*.

Lemma 58.A. If A is a deformation retract of X, then A has the same homotopy type as X.

Proof. Let $j : A \to X$ be the inclusion map and let $r : X \to A$ be the retraction mapping. Then the composite map $r \circ j$ equals the identity map on A. The composite map $j \circ r$ is homotopic to the identity map on X (since by the definition of deformation retract, there is a homotopy $H : X \times I \to X$ such that $H(x, 0) = x$

[the identity], $H(x,1) = r(x)$ [the retraction], and H is a homotopy between the identity map of X and the map $j \circ r$). So r and j are homotopy equivalences and spaces X and A have the same homotopy type. ∎

Note. We now show that Theorem 58.3 can be generalized from considering a space A that is a deformation retract of space X to two spaces which are of the same homotopy type. That is, we show that two spaces of the same homotopy type have isomorphic fundamental groups (in Theorem 58.7).

Lemma 58.4. Let $h, k : X \to Y$ be continuous maps. Let $h(x_0) = y_0$ and $k(x_0) = y_1$. If h and k are homotopic, there is a path α in Y from y_0 to y_1 such that $k_* = \hat{\alpha} \circ h_*$ (where k_* and h_* are induced homomorphisms on the fundamental group and $\hat{\alpha} : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ is defined as $\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha]$—see Figure 52.1). Indeed, if $H : X \times I \to Y$ is the homotopy between h and k, then α is the path $\alpha(t) = H(x_0, t)$.

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\alpha}"] \\ & \pi_1(Y, y_1) \end{tikzcd}$$

Corollary 58.5. Let $h, k : X \to Y$ be homotopic continuous maps. Let $h(x_0) = y_0$ and $k(x_0) = y_1$. If induced homomorphism h_* is injective (one to one), surjective (onto), or trivial then so is the induced homomorphism k_*.

Corollary 58.6. Let $h : X \to Y$. If h is nulhomotopic, then h_* is the trivial homomorphism.

Theorem 58.7. Let $f : X \to Y$ be continuous. Let $f(x_0) = y_0$. If f is a homotopy equivalence, then induced homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

Note. We can rephrase Theorem 58.7 as: "If (X, x_0) and (Y, y_0) are of the same homotopy type then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$." This shows that the definition of homotopy equivalence in terms of composition of mappings required to be homotopic to identity maps is the correct approach to yield isomorphic fundamental groups.

Note. The following result classifies the conditions under which two spaces have the same homotopy type.

Theorem of Fox. Two topological spaces X and Y have the same homotopy type (and hence have isomorphic fundamental groups) if and only if they are homeomorphic to deformation retracts of a single space Z.

Note. The Theorem of Fox first appeared in "On Homotopy Type and Deformation Retracts," *Annals of Mathematics*, **44**(2), 40–50 (1943), by Ralph H. Fox. An elementary proof appears in Martin Fuchs' "A Note on Mapping Cylinders," *Michigan Mathematical Journal*, **18**, 289–290 (1974). A copy of Fuchs' paper can be found through Project Euclid at:

`http://projecteuclid.org/download/pdf_1/euclid.mmj/1029000735`
(accessed 12/28/2014).

By the way, Ralph H. Fox is coauthor (along with Richard H. Crowell) of the classical *Introduction to Knot Theory*, Ginn and Company Publishing (1963), which is still in print through Dover Publications. Martin Fuchs should not be confused with the more widely known Laszlo Fuchs who published (among other things) a two volume work on *Infinite Abelian Groups*.

Note. Fox's Theorem shows us (again, but this time directly) that since the figure eight and theta space are both deformation retracts of the doubly punctured plane, both have the same fundamental group.

Revised: 1/11/2018