Section 59. The Fundamental Group of S^n

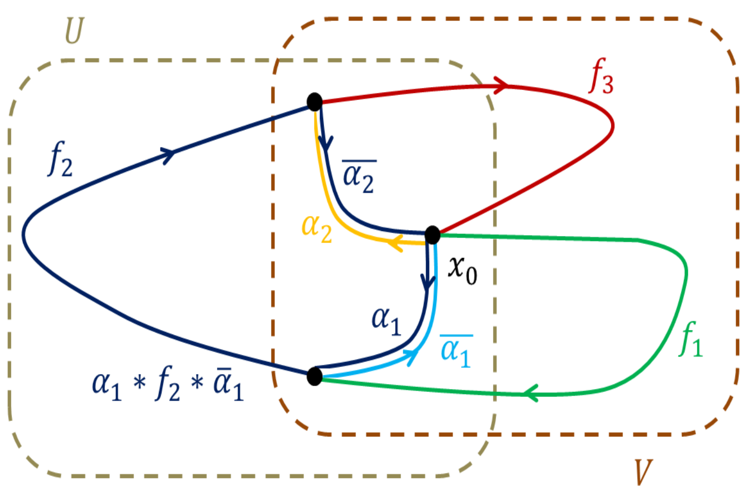
Note. In this section and the next, we show that the surfaces S^2, the torus, and the double torus are topologically distinct. We do so by showing that no two have the same fundamental group.

Theorem 59.1. Suppose $X = U \cup V$ where U and V are open sets of X. Suppose that $U \cap V$ is path connected and that $x_0 \in U \cap V$. Let i and j be the inclusion mappings of U and V, respectively, into X. Then the images of the induced homomorphisms

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \text{ and } j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

generate the group $\pi_1(X, x_0)$.

Note. The construction in Step 2 of the proof of Theorem 59.1 is illustrated here for $n = 3$:



Note. Theorem 59.1 is a special case of the Seifert-van Kampen Theorem which expresses the fundamental group of the space $X = U \cup V$, where $U \cap V$ is path connected, in terms of the fundamental groups of U and V (see Section 70 of Chapter 11). The Seifert-van Kampen Theorem is used in the classification of surfaces (see Chapter 12).

Corollary 59.2. Suppose $X = U \cap V$ where U and V are open sets of X. Suppose $U \cap V$ is nonempty and path connected. If U and V are simply connected then X is simply connected.

Note. Now for the main result of this section. Notice that the following theorem implies that the fundamental group of S^n is the trivial group for $n \geq 2$.

Theorem 59.3. If $n \geq 2$, the n-sphere is simply connected.

Note. The famous Poincaré Conjecture relates to simple connectivity and the 3-sphere. It states (in its original form) that: "Every simply connected closed three manifold is homeomorphic to the three sphere S^3." This was first conjectured by Henri Poincaré in 1904. A more general version was eventually stated as: "Every compact n-manifold is homotopy-equivalent to the n-sphere if and only if it is homeomorphic to the n sphere." In 2002 and 2003, the Russian Grigori Perelman proved the more general Thurston's Geometrization Conjecture which implies the Poincaré Conjecture. In recognition of his proof of the 100 year old conjecture, Perelman was offered the Fields Medal in 2006 and the one million dollar Millennium Prize by the Clay Mathematics Institute in 2010. He declined both prizes. Source: `http://mathworld.wolfram.com/PoincareConjecture.html` and Wikipedia (accessed 12/14/2014).

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