Section 60. Fundamental Groups of Some Surfaces

Note. In Section 36 of Chapter 4 we defined an *m-manifold* as a Hausdorff space X with a countable basis such that each point $x \in X$ has a neighborhood that is homeomorphic with an open subset of $\mathbb{R}^m$. A 1-manifold is a *curve* and a 2-manifold is a *surface*. In this section, we consider the fundamental groups of the projective plane, the torus, and the double torus. This will allow us to show that these surfaces and S^2 are all topologically distinct.

Note. The 2-sphere S^2, torus, and projective plane play very fundamental roles in the classification of all compact surfaces (up to homeomorphism). This classification is accomplished in Chapter 12.

Note. Recall that if $\langle G, * \rangle$ and $\langle G', *' \rangle$ are groups, then the Cartesian product $G \times G'$ is a group under the binary operation

$$(a, a') \cdot (b, b') = (a * b) \times (a' *' b').$$

If $h : C \to A$ and $k : C \to B$ are group homomorphisms then $\Phi : C \to A \times B$ defined as $\Phi(c) = h(c) \times k(c)$ is a group homomorphism. The following result will allow us to deal with the torus $S^1 \times S^1$.

Theorem 60.1. Let X and Y be topological spaces and $x_0 \in X$, $y_0 \in Y$. The group $\pi_1(X \times Y, x_0 \times y_0)$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Note. Since the fundamental group of S^1 is $\mathbb{Z}$ by Theorem 59.3, then Theorem 60.1 gives the following.

Corollary 60.2. The fundamental group of the torus $T = S^1 \times S^1$ is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.

Note. We have now established the profound result that S^2 and T are not homeomorphic, since they have different fundamental groups.

Definition. The *projective plane* P^2 is the quotient space obtained from S^2 by identifying each point $\vec{x} \in S^2$ with its antipodal point $-\vec{x}$. More precisely, P^2 consists of equivalence classes of points of S^2 under the equivalence relation $\vec{x} \sim \vec{y}$ if and only if $\vec{y} = \pm\vec{x}$.

Note. Munkres claims that the projective plane P^2 cannot be embedded in $\mathbb{R}^3$ (and hence is hard to visualize). In fact, the projective plane is a "nonorientable" surface. The following result shows that it is a surface.

Note. We need a model of P^2 to help us visualize its behavior. One way to do this is to take the closed upper half of S^2 and to say that the points along the boundary (the "equator") correspond when they are antipodal on S^2. An alternate way is to partition S^2 into two pieces where no two anitpodal points lie in the same set.

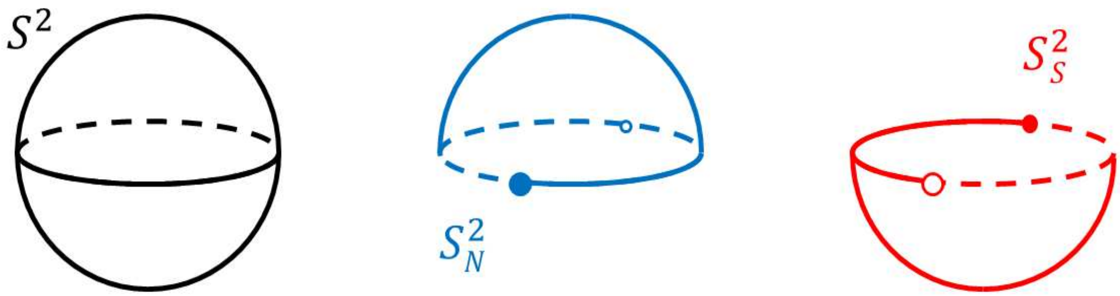
With S^2 represented as

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

define

$$S^2_N = \{\vec{x} \in S^2 \mid z > 0\} \cup \{(x, y, 0) \mid y > 0\} \cup \{(1, 0, 0)\}$$

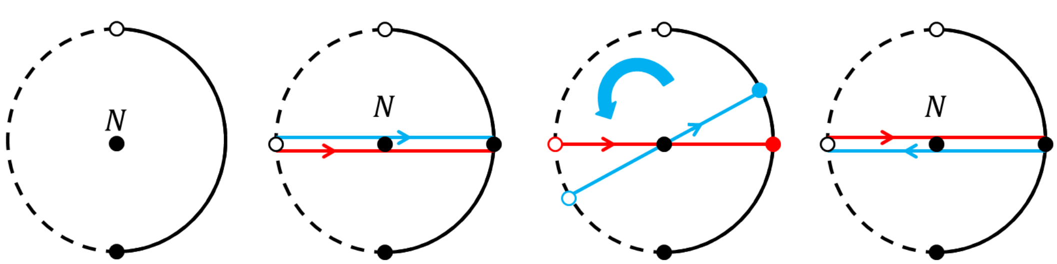
and $S^2_S = S^2 \setminus S^2_N$:



Now on S^2_N, we imagine antipodal boundary points are coincident.

Notice that if we take a loop based at $N = (0, 0, 1)$ which contains all points on S^2_N with x-coordinate 0, then this loop cannot be homotopic to a constant. This is because any continuous mapping of the loop must still contain a boundary point of S^2_N and so contains points near antipodal points on the boundary. But a funny thing happens with two such loops.

Imagine S^2_N squashed down to a circle on $\mathbb{R}^2$:



Take two copies of a path based at N which runs left to right in the squashed version of S^2_N. Now (homotopically) rotate one copy about N through $180°$. This yields the reverse of the unrotated path and therefore the equivalence class containing these paths is its own inverse in $\pi_1(P^2, N)$. Along with the equivalence class of constant loops based at N, we see that $\pi_1(P^2, N)$ has a subgroup isomorphic to $\mathbb{Z}_2$. In fact, we will see in Corollary 60.4 that the fundamental group itself is isomorphic to $\mathbb{Z}_2$.

Theorem 60.3. The projective plane P^2 is a surface, and the quotient map $p : S^2 \to P^2$ defined as $p(\vec{x}) = [\vec{x}] = \{-\vec{x}, \vec{x}\}$ is a covering map.

Corollary 60.4. $\pi_1(P^2, y)$ is a group of order 2.

Note. Projective n-space P^n can be similarly defined by "identifying" $\vec{x}$ and $-\vec{x}$ for each $\vec{x}$ on the n-sphere S^n. The proofs given for Theorem 60.3 and Corollary 60.4 carry through to show that $\pi_1(P^n, y) \cong \mathbb{Z}_2$ for $n \geq 2$.

Note. We now study the double torus. The following lemma is a preliminary result in showing that the fundamental group of a double torus is not abelian. In the proofs of the next two results, we rely somewhat on pictures instead of computations.

Lemma 60.5. The fundamental group of the figure eight is not abelian.

Note. In Example 3 on page 372 of Section 58, the "theta space" is introduced and it is claimed (based on deformation retracts) that the fundamental group of the theta space is isomorphic to the fundamental group of the figure eight. In Example 1 on page 432 of Section 70, it is shown that the fundamental group of the theta space (and therefore, up to isomorphism, the figure eight) is the free product of two infinite cyclic groups; that is, a free group on two generators.

Theorem 60.6. The fundamental group of the double torus is not abelian.

Corollary 60.7. The 2-sphere, torus, projective plane, and double torus are topologically distinct.

Proof. The fundamental group of the 2-sphere is the trivial group since S^2 is simply connected by Theorem 59.3. The fundamental group of the torus is congruent to $\mathbb{Z} \times \mathbb{Z}$ by Corollary 60.2. The fundamental group of the projective plane is isomorphic to $\mathbb{Z}_2$ by Corollary 60.4. Each of these groups is abelian and distinct. Since the fundamental group of the double torus is not abelian by Theorem 60.6, then it is a group distinct from the others. By Corollary 52.5, homeomorphic spaces have isomorphic fundamental groups, so these four surfaces are topologically distinct. ∎

Revised: 1/11/2018