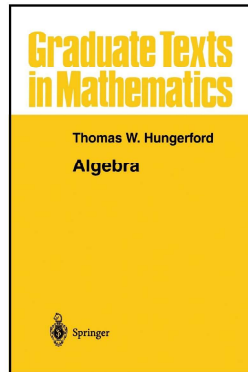


Modern Algebra

Chapter 0. Introduction: Prerequisites and Preliminaries

0.1–0.6. Logic, Sets and Classes, Functions, Relations and Partitions, Products, Integers—Proofs of Theorems



Corollary 0.3.A

Corollary 0.3.A. If $f : A \rightarrow B$, $g : B \rightarrow A$, and $h : B \rightarrow A$ where g is a left inverse of f and h is a right inverse of f . Then $g = h$. So a two-sided inverse of a function is unique.

Proof. We have

$$g = g1_B = g(fh) = (gf)h = 1_A h = h.$$

□

Corollary 0.3.B

Corollary 0.3.B. If A is a set and $f : A \rightarrow B$ then

f is bijective $\iff f$ has a two-sided inverse.

Proof. If f has a left inverse then by Theorem 0.3.1(i) f is one to one. If f has a right inverse then by Theorem 0.3.1(ii) f is onto. So if f has a two-sided inverse then f is a bijection.

If f is a bijection then by Theorem 0.3.1(i) f has a left inverse and by Theorem 0.3.1(ii) f has a right inverse. By the previous Corollary, f has a two-sided inverse. □

Lemma 0.4.A

Lemma 0.4.A. For $a, b \in A$ and R an equivalence relation on $A \times A$, we have either (1) $\bar{a} \cap \bar{b} = \emptyset$, or (2) $\bar{a} = \bar{b}$.

Proof. If $\bar{a} \cap \bar{b} = \emptyset$ then there is $c \in \bar{a} \cap \bar{b}$. Hence $c \sim a$ and $c \sim b$. By symmetry and transitivity, $a \sim b$. So $a \in \bar{b}$. Similarly, for any $d \sim a$ we have $d \sim b$ and so $d \in \bar{b}$, or more generally $\bar{a} \subset \bar{b}$. For any $e \sim b$ we likewise have $e \in \bar{a}$ and so $\bar{b} \subset \bar{a}$, and $\bar{a} = \bar{b}$. □